

NBER WORKING PAPER SERIES

DYNAMIC SPATIAL GENERAL EQUILIBRIUM

Benny Kleinman
Ernest Liu
Stephen J. Redding

Working Paper 29101
<http://www.nber.org/papers/w29101>

NATIONAL BUREAU OF ECONOMIC RESEARCH

1050 Massachusetts Avenue
Cambridge, MA 02138
July 2021, Revised October 2021

We are grateful to Princeton University for research support. A previous version of this paper circulated under the title “Sufficient Statistics for Dynamic Spatial Economics.” We would like to thank Treb Allen, Lorenzo Caliendo, Michael Peters, Ricardo Reyes-Heroles, Daniel Xu, and conference and seminar participants at CEPR ERWIT, Dallas, Dartmouth, Hong Kong, NBER Summer Institute, Nottingham, Princeton, the Virtual Trade and Macro Seminar, Urban Economics Association, and EAGLS for helpful comments. We would like to thank Maximilian Schwarz for excellent research assistance. The usual disclaimer applies. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

NBER working papers are circulated for discussion and comment purposes. They have not been peer-reviewed or been subject to the review by the NBER Board of Directors that accompanies official NBER publications.

© 2021 by Benny Kleinman, Ernest Liu, and Stephen J. Redding. All rights reserved. Short sections of text, not to exceed two paragraphs, may be quoted without explicit permission provided that full credit, including © notice, is given to the source.

Dynamic Spatial General Equilibrium
Benny Kleinman, Ernest Liu, and Stephen J. Redding
NBER Working Paper No. 29101
July 2021, Revised October 2021
JEL No. F14,F15,R12

ABSTRACT

We develop a dynamic spatial model with forward-looking investment and migration. We characterize the existence and uniqueness of the steady-state equilibrium; generalize existing dynamic exact-hat algebra techniques to incorporate investment; and linearize the model to provide an analytical characterization of the economy's transition path using spectral analysis. We show that U.S. states are closer to steady-state at the end of our sample period in 2015 than during the prior five decades. We find that much of the observed decline in the rate of income convergence across US states is explained by gradual adjustment given initial conditions, rather than by shocks to fundamentals, and that both capital and labor dynamics contribute to this gradual adjustment. We show that capital and labor dynamics interact with one another to generate slow and heterogeneous rates of convergence to steady-state.

Benny Kleinman
Department of Economics
Julis Romo Rabinowitz Building
Princeton, NJ 08544
binyamin@princeton.edu

Ernest Liu
Princeton University
Department of Economics
Julis Romo Rabinowitz Building
Princeton, NJ 08544
ernestliu@princeton.edu

Stephen J. Redding
Department of Economics &
School of Public and International Affairs
Princeton University
Princeton, NJ 08544
and CEPR
and also NBER
reddings@princeton.edu

1 Introduction

A central research question in economics is understanding the gradual response of the spatial distribution of economic activity to fundamental shocks, such as changes in productivity. In general, this response can be gradual, because of migration frictions for mobile factors (labor), and the gradual accumulation of immobile factors (capital structures). However, a key challenge has been modelling forward-looking capital investments in economic geography models with population mobility, because investment and migration decisions in each location depend on *one another*, and on these decisions in *all locations in all future time periods*.

We make four main contributions to addressing this research question. First, we develop a tractable dynamic spatial model with forward-looking investment and migration, and provide an analytical characterization of the existence and uniqueness of the steady-state equilibrium. Second, we generalize existing dynamic exact-hat algebra results for dynamic discrete choice migration models to our model with forward-looking investment. We are thus able to solve counterfactuals for changes in the model’s endogenous variables using only the observed values of endogenous variables in an initial equilibrium and assumed fundamental shocks, without having to solve for the unobserved initial levels of these fundamentals. Third, we linearize the model to provide an analytical characterization of the economy’s transition path. We show that the speed of convergence to steady-state depends on the spectral properties (the entire spectrum of eigencomponents) of a transition matrix that governs the updating of the labor and capital state variables over time. Fourth, we use our framework to understand the observed decline in rates of income convergence across U.S. states since the early 1960s. Our framework is well suited for this empirical application, because it allows for roles of initial conditions and shocks to fundamentals, and both capital and labor dynamics shape the speed of convergence to steady-state.

We begin with a baseline single-sector Armington model of trade, with dynamic discrete choice migration decisions, and investment determined as the solution to an intertemporal utility maximization problem. The economy consists of many locations that differ in productivity, amenities, and bilateral trade and migration costs. There are two types of agents: workers and landlords. Workers are geographically mobile but do not have access to an investment technology (and hence live “hand to mouth”). They make forward-looking migration decisions, taking into account migration costs and the expected continuation value from optimal future location decisions, as in [Caliendo et al. \(2019\)](#). Landlords are geographically immobile but have access to an investment technology for accumulating local capital. They make forward-looking consumption-investment decisions, as in [Angeletos \(2007\)](#) and [Moll \(2014\)](#).

Although our framework allows for a high-dimensional state space with many locations that can differ in productivity and amenities, and incorporates a rich geography of bilateral trade and

migration costs, we derive analytical conditions for the existence and uniqueness of the steady-state equilibrium. We show that these conditions depend only on structural parameters of the model, such as agglomeration and dispersion forces, and are invariant with respect to the initial spatial distribution of economic activity. Given the observed values of the endogenous variables for an initial equilibrium somewhere along the transition path towards an unobserved steady-state, we use dynamic exact-hat algebra techniques to solve for the economy's transition path for any sequence of future changes in fundamentals. An implication of this result is that we can invert the model to solve for unobserved changes in productivity, amenities, trade costs and migration costs, using only the observed values of the endogenous variables, and without making assumptions about where the economy lies along the transition path.

We show that the economy's transition dynamics are shaped by a powerful interaction between investment and migration decisions. Convergence towards steady-state is slow when the gaps of capital and labor from steady-state are positively correlated across locations, such that capital and labor in each location tend to either be both above or both below steady-state. In contrast convergence towards steady-state is fast when these gaps are negatively correlated across locations, such that capital tends to be above steady-state when labor is below steady-state, or vice versa. The reason is the complementary between capital and labor in the production technology. When capital is above steady-state, this raises the marginal productivity of labor, which dampens the downward adjustment of labor and migration to other locations. Similarly, when labor is above steady-state, this raises the marginal productivity of capital, which retards the downwards adjustment of capital. A key takeaway from these findings is that the interaction between capital accumulation and migration is central to understanding the persistent and heterogeneous impacts of local labor market shocks.

To further understand this interaction, we linearize the model to obtain a closed-form solution for the economy's transition path in terms of an impact matrix, which captures the initial impact of shocks to fundamentals on the state variables, and a transition matrix, which governs the updating of the state variables from one period to the next. We show that both the speed of convergence towards steady-state and the evolution of the state variables along the transition path depend on the spectral properties of this transition matrix. We introduce the concept of an eigen-shock: a shock to productivity and amenities for which the initial impact on the capital and labor state variables corresponds to an eigenvector of the transition matrix. We show that the speed of convergence to steady-state for an eigen-shock depends solely on the associated eigenvalue of the transition matrix. Each eigen-shock has a different speed of convergence, because it involves a different incidence of productivity and amenity shocks across locations, which in turn implies a different initial impact on the labor and capital state variables in each location. We can solve for this transition matrix from the observed data, and hence recover its eigenval-

ues and eigenvectors, and the eigen-shocks. We show that any empirical shocks to productivity and amenities can be written as linear combinations of these eigen-shocks. Furthermore, the weights in these linear combinations can be recovered from linear projections (regressions) of the empirical shocks on the eigen-shocks.

We implement our quantitative framework using U.S. state-level data for the period from 1965-2015, including income, capital, population, bilateral shipments of goods and bilateral migration flows. We find that capital dynamics differ substantially from population dynamics, highlighting the relevance of a framework such as ours that incorporates capital accumulation. We demonstrate substantial net and gross migration between states, confirming the importance of incorporating these migration flows. We confirm that both bilateral goods and migration flows are well approximated by gravity equations in origin characteristics, destination characteristics, and proxies for bilateral frictions, as implied by our theoretical framework.

We find slow rates of convergence to steady-state, such that individual U.S. states can remain persistently above or below steady-state for decades. As early as the 1960s, the Rust Belt states of the North-East and Mid-West were substantially above their steady-state, whereas the Sun Belt states of the South and West were markedly below their steady-state. By the end of our sample period in 2015, we find that U.S. states are closer to their steady-states than they were during the prior five decades. As a result, much of the observed decline in the rate of regional convergence of income per capita since the early 1960s is explained by initial conditions, rather than by shocks to fundamentals. We find that both capital accumulation and migration play an important role in this process of gradual adjustment. Of the two, capital accumulation is more important for the convergence in income per capita, whereas migration is required to match the observed changes in the population distribution across U.S. states.

We find that the empirical productivity or amenity shocks from our model inversion exhibit non-negligible weights throughout the spectrum of eigen-shocks, since these empirical shocks affect many locations simultaneously. Therefore, the speed of convergence in response to these empirical shocks depends on the entire spectrum of eigencomponents. We show that our findings of slow convergence to steady-state are explained by the positive correlation across locations of the empirical gaps of labor and capital from steady-state. This positive correlation in turn reflects the impact of both initial conditions and fundamental shocks. We find that the correlation across locations between the empirical productivity and amenity shocks is positive up to 1975, which tends to increase the gap of both labor and capital from steady-state. From 1975 onwards, we find that this correlation between the shocks becomes negative, before diminishing to close to zero by the end of our sample period. Consistent with this pattern of results, we find that the explanatory power of predicted population growth based on convergence towards an initial steady-state with unchanged fundamentals is particularly strong from 1975 onwards.

We show that the tractability of our approach lends itself to a large number of extensions and generalizations. We incorporate agglomeration forces in both production and residential decisions. We show that our results hold for an entire class of constant elasticity trade models, including models of perfect competition and constant returns to scale, and models of monopolistic competition and increasing returns to scale. We generalize our approach to incorporate residential capital use (housing) and to allow landlords to invest in other locations. We extend our analysis to incorporate multiple sectors and input-output linkages.

Our research is related to several strands of existing work. First, our paper contributes to a long line of research on economic geography, including [Krugman \(1991\)](#), [Krugman and Venables \(1995\)](#) and [Helpman \(1998\)](#), as synthesized in [Fujita et al. \(1999\)](#), and reviewed in [Duranton and Puga \(2004\)](#) and [Redding \(2020\)](#). Early theoretical research in this area considered static models. More recent research on quantitative spatial models also has typically focused on such static specifications, including [Redding and Sturm \(2008\)](#), [Allen and Arkolakis \(2014\)](#), [Ahlfeldt et al. \(2015\)](#), [Allen et al. \(2017\)](#), [Ramondo et al. \(2016\)](#), [Redding \(2016\)](#), [Caliendo et al. \(2018\)](#) and [Monte et al. \(2018\)](#), as surveyed in [Redding and Rossi-Hansberg \(2017\)](#).

Second, the challenge of modelling of the interaction between forward-looking decisions for both investment and migration has led existing economic geography research to consider specifications in which dynamic decisions reduce to static problems. In the innovation models of [Desmet and Rossi-Hansberg \(2014\)](#), [Desmet et al. \(2018\)](#) and [Peters \(2019\)](#), the incentive to invest in innovation each period depends on the comparison of static profits and innovation costs. In the overlapping generations model of [Allen and Donaldson \(2020\)](#), adults make migration decisions to maximize their own adult utility, and do not consider the utility of the next generation of youths. Another approach is to capture forward-looking migration decisions using dynamic discrete choice models, including [Artuç et al. \(2010\)](#), [Caliendo et al. \(2019\)](#) and [Caliendo and Parro \(2020\)](#).¹ Our tractable framework incorporates forward-looking decisions for both migration and investment, and shows that the interaction between these two sources of dynamics is central to the persistent and heterogeneous impact of local labor market shocks.

Third, our research connects with dynamic models of capital accumulation and international trade, including [Anderson et al. \(2015\)](#), [Eaton et al. \(2016\)](#), [Alvarez \(2017\)](#) and [Ravikumar et al. \(2019\)](#). While each of these papers introduces forward-looking investment decisions, a key difference from our economic geography setting is that labor is assumed to be immobile across countries in international trade models. Our key contribution is to develop a tractable dynamic spatial model that incorporates forward-looking decisions for both migration and investment. In our analytical characterization of capital and labor dynamics, we use techniques from the Dy-

¹See [Glaeser and Gyourko \(2005\)](#) and [Greaney \(2020\)](#) for models in which population dynamics are shaped by durable housing. See [Walsh \(2019\)](#) for a model in which innovation takes the form of the creation of new varieties.

dynamic Stochastic General Equilibrium (DSGE) literature following [Blanchard and Kahn \(1980\)](#) and [Uhlig \(1999\)](#). In this literature, the convergence properties of the model typically depend only on the eigenvalue with the largest absolute value, because this determines the dynamic properties of the system over long time horizons. In contrast, in our dynamic spatial model, the dynamics of the spatial distribution of economic activity depends on the entire spectrum of eigenvalues and eigenvectors, because the eigen-shocks provide a basis for the space of possible productivity and amenity shocks. Relative this literature, we also consider a much higher-dimensional state space, with many locations or location-sectors.

Fourth, our paper is related to research on regional convergence, including [Barro and Sala-i-Martin \(1992\)](#), [Kim \(1995\)](#), [Mitchener and McLean \(1999\)](#) and [Ganong and Shoag \(2017\)](#). While early studies found a strong negative relationship between the rate of growth and initial level of income per capita (β -convergence), more recent research has documented a decline in rates of income convergence over time. As in closed-economy growth models, our framework incorporates capital accumulation as a key force for regional income convergence. Unlike these closed-economy growth models, we incorporate bilateral migration and goods trade as two important additional forces that shape regional income convergence across U.S. states.

Finally, our work also connects with the empirical literature that has established persistent impacts of local labor market shocks, including [Blanchard and Katz \(1992\)](#), [Autor et al. \(2013\)](#) and [Dix-Carneiro and Kovak \(2017\)](#). For example, [Dix-Carneiro and Kovak 2017](#) provides empirical evidence that Brazil's trade liberalization in the late 1980s continued to affect local labor market outcomes for more than 20 years afterwards, because of the downward adjustment in complementary capital investments. Our dynamic spatial model provides theoretical microfoundations for these empirical findings: the outmigration of workers from regions experiencing negative shocks induces a gradual decline in the complementary capital stock, which in turn induces a further outmigration of workers.

The remainder of the paper is structured as follows. In [Section 2](#), we introduce our baseline quantitative spatial model. We characterize the existence and uniqueness of the steady-state equilibrium and generalize existing dynamic exact-hat algebra approaches to solve for the transition path of the full non-linear model. In [Section 3](#), we linearize the model, and provide an analytical characterization of the economy's transition path using spectral analysis. In [Section 4](#), we show that our framework admits many extensions, including agglomeration forces, multiple sectors, and input-output linkages. In [Section 5](#), we implement our baseline single-sector specification for U.S. states from 1965-2015, and our multi-sector extension for the shorter period from 1999-2015 for which sectoral data are available. In [Section 6](#), we summarize our conclusions.

2 Dynamic Spatial Model

In this section, we introduce our baseline dynamic spatial general equilibrium model. We combine a specification of trade between locations with a constant trade elasticity, a formulation of migration decisions with a constant migration elasticity, optimal consumption-investment decisions with constant intertemporal elasticity of substitution preferences, and rational expectations about the future sequence of shocks to fundamentals.

We consider an economy with many locations indexed by $i \in \{1, \dots, N\}$. Time is discrete and is indexed by t . There are two types of infinitely-lived agents: workers and landlords. Workers are endowed with one unit of labor that is supplied inelastically and are geographically mobile subject to migration costs. Workers do not have access to an investment technology, and hence live “hand to mouth,” as in [Kaplan and Violante \(2014\)](#). Landlords are geographically immobile and own the capital stock in their location. They make forward-looking decisions over consumption and investment in this local stock of capital. We interpret capital as buildings and structures, which are geographically immobile once installed, and depreciate gradually at a constant rate δ .

The endogenous state variables are the population (ℓ_{it}) and capital stock (k_{it}) in each location. The key location characteristics that determine the spatial distribution of economic activity are the sequences of productivity (z_{it}), amenities (b_{it}), bilateral trade costs (τ_{nit}) and bilateral migration costs (κ_{nit}). Without loss of generality, we normalize the total population across all locations to one ($\sum_{i=1}^N \ell_{it} = 1$), so that ℓ_{it} can also be interpreted as the population share of location i at time t . Throughout the following, we use bold math font to denote a vector (lowercase letters) or matrix (uppercase letters). The derivations for all expressions and results in this section are reported in Section B of the online appendix.

To streamline the exposition, we focus in this section on shocks to productivities (z_{it}) and amenities (b_{it}), assuming that productivity and amenities are exogenous. In Section 4 below, we show that our approach extends to shocks to trade and migration costs, endogenous productivity and amenities through agglomeration forces, and other generalizations.

2.1 Production

At the beginning of each period t , the economy inherits in each location i a mass of workers (ℓ_{it}) and capital stock (k_{it}). Firms in each location use labor and capital to produce output (y_{it}) of the variety supplied by that location. Production is assumed to occur under conditions of perfect competition and subject to the following constant returns to scale technology:

$$y_{it} = z_{it} \left(\frac{\ell_{it}}{\mu} \right)^\mu \left(\frac{k_{it}}{1 - \mu} \right)^{1 - \mu}, \quad 0 < \mu < 1, \quad (1)$$

where z_{it} denotes productivity in location i at time t .

We assume that trade between locations is subject to iceberg variable trade costs, such that $\tau_{nit} \geq 1$ units of a good must be shipped from location i in order for one unit to arrive in location n , where $\tau_{nit} > 1$ for $n \neq i$ and $\tau_{iit} = 1$. From profit maximization, the cost to a consumer in location n of sourcing the good produced by location i depends solely on these iceberg trade costs and constant marginal costs:

$$p_{nit} = \tau_{nit} p_{iit} = \frac{\tau_{nit} w_{it}^\mu r_{it}^{1-\mu}}{z_{it}}, \quad (2)$$

where p_{iit} is the “free on board” price of the good supplied by location i before trade costs.

We choose the total labor income of all locations as our numeraire: $\sum_{i=1}^N w_{it} \ell_{it} = 1$.

2.2 Worker Consumption

Worker preferences within each period t are modeled as in the standard Armington model of trade with constant elasticity of substitution (CES) preferences. As workers do not have access to an investment technology, they spend their wage income and choose their consumption of varieties to maximize their utility each period. The indirect utility function of a worker in location n in period t depends on the worker’s wage (w_{nt}), the cost of living (p_{nt}) and amenities (b_{nt}):

$$u_{nt}^w = \frac{b_{nt} w_{nt}}{p_{nt}}, \quad (3)$$

where we use the superscript w to denote workers. The cost of living (p_{nt}) depends on the price of the variety sourced from each location i (p_{nit}):

$$p_{nt} = \left[\sum_{i=1}^N p_{nit}^{-\theta} \right]^{-1/\theta}, \quad (4)$$

where $\sigma > 1$ is the constant elasticity of substitution and $\theta = \sigma - 1 > 0$ is the trade elasticity.

2.3 Capital Accumulation

Landlords in each location choose their consumption and investment to maximize their intertemporal utility subject to their budget constraint. Landlords’ intertemporal utility equals the expected present discounted value of their flow utility:

$$v_{it}^k = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^{t+s} \frac{(c_{it+s}^k)^{1-1/\psi}}{1-1/\psi}, \quad (5)$$

where we use the superscript k to denote landlords; c_{it}^k is the consumption index dual to the price index (4); β is the discount rate; ψ is the elasticity of intertemporal substitution. Since landlords

are geographically immobile, we omit the term in amenities from their flow utility, because this does not affect the equilibrium in any way, and hence is without loss of generality.

We assume that the investment technology in each location uses the varieties from all locations with the same functional form as consumption. Landlords can produce one unit of capital in their location using one unit of the consumption index in that location.² We interpret capital as buildings and structures, which are geographically immobile once installed. Capital is assumed to depreciate at the constant rate δ and we allow for the possibility of negative investment.

The intertemporal budget constraint for landlords in each location requires that total income from the existing stock of capital ($r_{it}k_{it}$) equals the total value of their consumption ($p_{it}c_{it}^k$) plus the total value of net investment ($p_{it}(k_{it+1} - (1 - \delta)k_{it})$):

$$r_{it}k_{it} = p_{it}(c_{it}^k + k_{it+1} - (1 - \delta)k_{it}). \quad (6)$$

We begin by establishing a key property of landlords' optimal consumption-investment decisions. We use $R_{it} \equiv 1 - \delta + r_{it}/p_{it}$ to denote the gross return on capital.

Lemma 1. *The optimal consumption of location i 's landlords satisfies $c_{it} = \varsigma_{it}R_{it}k_{it}$, where ς_{it} is defined recursively as*

$$\varsigma_{it}^{-1} = 1 + \beta^\psi \left(\mathbb{E}_t \left[R_{it+1}^{\frac{\psi-1}{\psi}} \varsigma_{t+1}^{-\frac{1}{\psi}} \right] \right)^\psi.$$

Landlords' optimal saving and investment decisions satisfy $k_{it+1} = (1 - \varsigma_{it})R_{it}k_{it}$.

Lemma 1 shows that the landlords in each location have a linear saving rate $(1 - \varsigma_{it})$ out of current period wealth $R_{it}k_{it}$, as in Angeletos (2007). The saving rate depends on the expectation over the sequence of future returns on capital $\{R_{it+s}\}$, the discount rate β , and the elasticity of substitution ψ . In the special case of log-utility ($\psi = 1$), landlords have a constant saving rate β (i.e., $k_{it+1} = \beta R_{it}k_{it}$) as in the conventional Solow-Swan model and Moll (2014).

We show below that there exists a steady-state equilibrium level of the capital-labor ratio in each location, towards which the economy gradually converges in the absence of further shocks. In this steady-state equilibrium, the real rental rate in terms of the consumption good is equalized across locations.³ Along the transition path, the rental rate on capital can differ across locations, which is consistent with different rates of return to investments in immobile buildings and structures in expanding locations (such as San Francisco) than in declining locations (such as Detroit).

²Although we make the standard assumption that consumption and investment use goods in the same proportions, similar results hold if investment uses only the good produced by each location.

³In steady-state: $r_i^*/p_i^* = (1 - \beta(1 - \delta))/\beta$. Steady-state differences in r_i^*/p_i^* can be accommodated by differences in the productivity of investment, where one unit capital in location i is produced with θ_i units of the consumption good in that location.

Nevertheless, the extent of these differences is limited by the use of tradeable consumption goods for investment, and quantitatively we find them to be small.⁴

Our baseline specification makes two simplifying assumptions for tractability. First, we separate the capital accumulation and migration decisions by assuming that they are made by landlords and workers, respectively. Nevertheless, we show that these two forward-looking decisions interact in an important way through the marginal products of capital and labor in the production technology. Allowing workers to make consumption-investment decisions would introduce a further source of interaction, because workers' choice of location would be influenced by consumption smoothing over time.

Second, we assume that landlords can only invest in their own location. in Section D.8 of the online appendix, we develop an extension in which landlords can invest in other locations, and bilateral investment flows satisfy a gravity equation, as observed empirically. Again this extension introduces a further source of interaction between capital accumulation and migration. We abstract from this possibility in our baseline specification to focus on the interaction between these decisions through the production technology. Additionally, comprehensive data on bilateral investment flows between states are not available, and we find only small differences in the real return to capital along the transition path in our baseline specification.

2.4 Worker Migration Decisions

After supplying labor and spending wage income on consumption in each period t , workers observe idiosyncratic mobility shocks (ϵ_{gt}), and decide where to move. The value function for a worker in location i in period t (\mathbb{V}_{it}^w) is equal to the current flow of utility in that location plus the expected continuation value from the optimal choice of location:

$$\mathbb{V}_{it}^w = \ln u_{it}^w + \max_{\{g\}_1^N} \left\{ \beta \mathbb{E}_t [\mathbb{V}_{gt+1}^w] - \kappa_{git} + \rho \epsilon_{gt} \right\}, \quad (7)$$

where β is the discount rate; $\mathbb{E}_t [\cdot]$ denotes the expectation in period t over future location characteristics. We assume log utility for workers, because they live hand-to-mouth, and hence there is no role for the intertemporal elasticity of substitution in their consumption decisions. We make the conventional assumption that idiosyncratic mobility shocks are drawn from an extreme value distribution: $F(\epsilon) = e^{-e^{(-\epsilon - \bar{\gamma})}}$, where $\bar{\gamma}$ is the Euler-Mascheroni constant; the parameter ρ controls the dispersion of idiosyncratic mobility shocks; and we assume that bilateral migration costs satisfy $\kappa_{iit} = 1$ and $\kappa_{nit} > 1$ for $n \neq i$.

⁴It is straightforward to introduce another type of capital (e.g. machines) that is perfectly mobile across locations.

2.5 Market Clearing

Goods market clearing implies that income in each location, which equals the sum of the income of workers and landlords, is equal to expenditure on the goods produced by that location:

$$(w_{it}\ell_{it} + r_{it}k_{it}) = \sum_{n=1}^N S_{nit} (w_{nt}\ell_{nt} + r_{nt}k_{nt}), \quad (8)$$

where we begin by assuming that trade is balanced, before later extending our analysis to incorporate trade imbalances in Section 4.

Capital market clearing implies that the rental rate for capital is determined by the requirement that landlords' income from the ownership of capital equals payments for its use. Using profit maximization and zero profits, this capital market clearing condition is given by:

$$r_{it}k_{it} = \frac{1 - \mu}{\mu} w_{it}\ell_{it}. \quad (9)$$

2.6 General Equilibrium

Given the state variables $\{\ell_{i0}, k_{i0}\}$, the general equilibrium of the economy is the stochastic process of allocations and prices such that firms in each location choose inputs to maximize profits, workers make consumption and migration decisions to maximize utility, landlords make consumption and investment decisions to maximize utility, and prices clear all markets, with the appropriate measurability constraint with respect to the realizations of location fundamentals. For expositional clarity, we collect the equilibrium conditions and express them in terms of a sequence of five endogenous variables $\{\ell_{it}, k_{it}, w_{it}, R_{it}, v_{it}\}_{t=0}^{\infty}$. All other endogenous variables of the model can be recovered as a function of these variables.

Capital Returns and Accumulation: Using capital market clearing (9), the gross return on capital in each location i must satisfy:

$$R_{it} = \left(1 - \delta + \frac{1 - \mu}{\mu} \frac{w_{it}\ell_{it}}{p_{it}k_{it}} \right),$$

where the price index (4) of each location becomes

$$p_{nt} = \left[\sum_{i=1}^N \left(w_{it} \left(\frac{1 - \mu}{\mu} \right)^{1-\mu} (\ell_{it}/k_{it})^{1-\mu} \tau_{ni}/z_i \right) \right]^{-\theta} \quad (10)$$

The law of motion for capital is

$$k_{it+1} = (1 - \varsigma_{it}) \left(1 - \delta + \frac{1 - \mu}{\mu} \frac{w_{it}\ell_{it}}{p_{it}k_{it}} \right) k_{it}, \quad (11)$$

where $(1 - \varsigma_{it})$ is the saving rate defined recursively as in Lemma 1:

$$\varsigma_{it}^{-1} = 1 + \beta^\psi \left(\mathbb{E}_t \left[R_{it+1}^{\frac{\psi-1}{\psi}} \varsigma_{t+1}^{-\frac{1}{\psi}} \right] \right)^\psi .$$

Goods Market Clearing: Using the CES expenditure share, the equilibrium pricing rule (2), and the capital market clearing condition (9) in the goods market clearing condition (8), the requirement that income equals expenditure on the goods produced by a location can be written solely in terms of labor income:

$$w_{it} \ell_{it} = \sum_{n=1}^N S_{nit} w_{nt} \ell_{nt}, \quad (12)$$

$$S_{nit} = \frac{(w_{it} (\ell_{it}/k_{it})^{1-\mu} \tau_{ni}/z_i)^{-\theta}}{\sum_{m=1}^N (w_{mt} (\ell_{mt}/k_{mt})^{1-\mu} \tau_{nm}/z_m)^{-\theta}}, \quad T_{int} \equiv \frac{S_{nit} w_{nt} \ell_{nt}}{w_{it} \ell_{it}}, \quad (13)$$

where we have used the property that capital income is a constant multiple of labor income; S_{nit} is the expenditure share of importer n on exporter i at time t ; we have defined T_{int} as the corresponding income share of exporter i from importer n at time t ; and note that the order of subscripts switches between the expenditure share (S_{nit}) and the income share (T_{int}), because the first and second subscripts will correspond below to rows and columns of a matrix, respectively.

Worker Value Function: Using the value function (7), indirect utility function (3) and the properties of the extreme value distribution, the expected value from living in location n at time t after taking expectations with respect to the idiosyncratic mobility shocks $\{\epsilon_{gt}\}$ (that is, $v_{nt}^w \equiv \mathbb{E}_\epsilon [\mathbb{V}_{nt}^w]$), can be written as:

$$v_{nt}^w = \ln b_{nt} + \ln \left(\frac{w_{nt}}{p_{nt}} \right) + \rho \ln \sum_{g=1}^N (\exp(\beta \mathbb{E}_t v_{gt+1}^w) / \kappa_{gnt})^{1/\rho}, \quad (14)$$

where the expectation $\mathbb{E}_t [v_{gt+1}^w] = \mathbb{E}_t \mathbb{E}_\epsilon [\mathbb{V}_{nt+1}^w]$ is taken over future fundamentals $\{z_{is}, b_{is}\}_{s=t+1}^\infty$.

Population Flow: Using the properties of the extreme value distribution, the population flow condition for the evolution of the population distribution over time is given by:

$$\ell_{gt+1} = \sum_{i=1}^N D_{igt} \ell_{it}, \quad (15)$$

$$D_{igt} = \frac{(\exp(\beta \mathbb{E}_t v_{gt+1}^w) / \kappa_{git})^{1/\rho}}{\sum_{m=1}^N (\exp(\beta \mathbb{E}_t v_{mt+1}^w) / \kappa_{mit})^{1/\rho}}, \quad E_{git} \equiv \frac{\ell_{it} D_{igt}}{\ell_{gt+1}}, \quad (16)$$

where D_{igt} is the outmigration probability from location i to location g between time t and $t + 1$; we have defined E_{git} as the corresponding immigration probability to location g from location i between time t and $t + 1$; again note that the order of subscripts switches between the outmigration probability (D_{igt}) and the immigration probability (E_{git}), because the first and second subscripts will correspond below to rows and columns of a matrix, respectively.

Properties of General Equilibrium: Given the state variables $\{\ell_{it}, k_{it}\}$ and the realized location fundamentals $\{z_{it}, b_{it}\}$, the general equilibrium in each period is determined as in a standard static international trade model. Between periods, the evolution of the stock of capital $\{k_{it}\}$ is determined by the equilibrium saving rate, and the dynamics of the population distribution $\{\ell_{it}\}$ are determined by the gravity equation for migration. We now formally define equilibrium.

Definition 1. Equilibrium. Given the state variables $\{\ell_{i0}, k_{i0}\}$ in each location in an initial period $t = 0$, an *equilibrium* is a stochastic process of wages, capital returns, expected values, mass of workers and stock of capital in each location $\{w_{it}, R_{it}, v_{it}, \ell_{it+1}, k_{it+1}\}_{t=0}^{\infty}$ measurable with respect to the fundamental shocks up to time t ($\{z_{is}, b_{is}\}_{s=1}^t$), and solves the value function (14), the population flow condition (15), the goods market clearing condition (12), and the capital market clearing and accumulation condition (11), with the saving rate determined by Lemma 1.

We define a deterministic steady-state equilibrium as one in which the fundamentals $\{z_i^*, b_i^*\}$ and the endogenous variables $\{\ell_i^*, k_i^*, w_i^*, R_i^*, v_i^*\}$ are constant over time, where we use an asterisk to denote the steady-state value of variables.

Definition 2. Steady State. A *steady-state* of the economy is an equilibrium in which all location-specific fundamentals and endogenous variables (wages, expected values, mass of workers and stock of capital in each location) are time invariant: $\{z_i^*, b_i^*, \ell_i^*, k_i^*, w_i^*, R_i^*, v_i^*\}$.

Our model features rich spatial interactions between locations in both goods and factor markets and forward-looking investment and migration decisions. Nevertheless, the absence of agglomeration forces and diminishing marginal returns to capital accumulation ensure the existence of a unique steady-state spatial distribution of economic activity (up to a choice of numeraire), given time-invariant values of the locational fundamentals of productivity (z_i^*), amenities (b_i^*), goods trade costs (τ_{ni}^*) and migration frictions (κ_{ni}^*).

Proposition 1. Existence and Uniqueness. *There exists a unique steady-state spatial distribution of economic activity $\{\ell_i^*, k_i^*, w_i^*, R_i^*, v_i^*\}$ (up to a choice of numeraire) given time-invariant locational fundamentals $\{z_i^*, b_i^*, \tau_{ni}^*, \kappa_{ni}^*\}$ that is independent of initial conditions $\{\ell_{i0}, k_{i0}\}$.*

Proof. See Section B.7 of the online appendix. □

Trade and Migration Share Matrices: We now introduce the trade and migration share matrices that we use to characterize the model's transition dynamics in response to shocks to fundamentals. Let \mathbf{S} be the $N \times N$ matrix with the ni -th element equal to importer n 's expenditure on exporter i . Let \mathbf{T} be the $N \times N$ matrix with the in -th element equal to the fraction of income that exporter i derives from selling to importer n . We refer to \mathbf{S} as the *expenditure share* matrix and to \mathbf{T} as the *income share* matrix. Intuitively, S_{ni} captures the importance of i as a supplier to location n , and T_{in} captures the importance of n as a buyer for country i . Note the order of subscripts: in matrix \mathbf{S} , rows are buyers and columns are suppliers, whereas in matrix \mathbf{T} , rows are suppliers and columns are buyers.

Similarly, let \mathbf{D} be the $N \times N$ matrix with the ni -th element equal to the share of outmigrants from origin n to destination i . Let \mathbf{E} be the $N \times N$ matrix with the in -th element equal to the share of immigrants to destination i from origin n . We refer to \mathbf{D} as the *outmigration* matrix and to \mathbf{E} as the *immigration* matrix. Intuitively, D_{ni} captures the importance of i as a destination for origin n , and E_{in} captures the importance of n as an origin for destination i . Again note the order of subscripts: in matrix \mathbf{D} , rows are origins and columns are destinations, whereas in matrix \mathbf{E} , rows are destinations and columns are origins.⁵

Dynamic Exact Hat Algebra We now generalize existing dynamic exact-hat algebra results for undertaking counterfactuals under perfect foresight in migration models from [Caliendo et al. \(2019\)](#) to incorporate forward-looking investment decisions. We suppose that we observe the spatial distribution of economic activity somewhere along the transition path towards an unobserved steady-state. Given the initial observed endogenous variables of the model, we show that we are able to solve for the economy's transition path in time differences ($\dot{x}_{t+1} = x_{t+1}/x_t$) for any anticipated convergent sequence of future changes in fundamentals, without having to solve for the unobserved initial level of fundamentals.

Proposition 2. Dynamic Exact-hat Algebra. *Given an initial observed allocation of the economy, $\left(\{l_{i0}\}_{i=1}^N, \{k_{i0}\}_{i=1}^N, \{k_{i1}\}_{i=1}^N, \{S_{ni0}\}_{n,i=1}^N, \{D_{ni,-1}\}_{n,i=1}^N\right)$, and a convergent sequence of future changes in fundamentals under perfect foresight, $\left\{\left\{\dot{z}_{it}\right\}_{i=1}^N, \left\{\dot{b}_{it}\right\}_{i=1}^N, \left\{\dot{\tau}_{ijt}\right\}_{i,j=1}^N, \left\{\dot{\kappa}_{ijt}\right\}_{i,j=1}^N\right\}_{t=1}^{\infty}$, the solution for the sequence of changes in the model's endogenous variables does not require information on the level of fundamentals, $\left\{\left\{z_{it}\right\}_{i=1}^N, \left\{b_{it}\right\}_{i=1}^N, \left\{\tau_{ijt}\right\}_{i,j=1}^N, \left\{\kappa_{ijt}\right\}_{i,j=1}^N\right\}_{t=0}^{\infty}$.*

⁵For theoretical completeness, we maintain three assumptions on these matrices, which are satisfied empirically in all years of our data: **(i)** For any i, n , there exists k such that $[\mathbf{S}^k]_{in} > 0$ and $[\mathbf{D}^k]_{in} > 0$. **(ii)** For all i , $\mathbf{S}_{ii} > 0$ and $\mathbf{D}_{ii} > 0$. **(iii)** The matrices \mathbf{S} and \mathbf{D} are of rank $N - 1$. The first assumption states that all locations are connected with each other directly or indirectly, through flows of goods and migrants. The second assumption ensures that each location consumes a positive amount of domestic goods and has a positive amount of own migrants. The third assumption ensures that $N - 1$ columns of the trade and migration share matrices are linearly independent, with the final column determined by the requirement that the trade and migration shares sum to one.

Proof. See Section B.8 of the online appendix. □

Intuitively, we use the initial observed endogenous variables and the equilibrium conditions of the model to control for the unobserved initial level of fundamentals. From this proposition, we can use dynamic exact-hat algebra methods to solve for the unobserved initial steady-state in the absence of any further changes in fundamentals.⁶ We can also use this approach to solve counterfactuals for the transition path of the spatial distribution of economic activity in response to assumed sequences of future changes in fundamentals.

A corollary of these dynamic exact-hat algebra results in Proposition 2 is that we can invert the model to solve for the unobserved changes in productivity, amenities, trade costs and migration costs that are implied by the observed changes in the endogenous variables of the model under perfect foresight, as shown in Section B.9 of the online appendix. Importantly, we can undertake this model inversion along the transition path without making assumptions about the precise sequence of future fundamentals, because the observed changes in migration flows and the capital stock capture agents' expectations about this sequence of future fundamentals.

3 Spectral Analysis

To further understand the role of capital and labor dynamics, we now linearize the model to provide an analytical characterization of the economy's transition path using spectral analysis. Throughout this section, we focus for expositional convenience on shocks to productivity and amenities, but show that our approach generalizes to incorporate shocks to migration and trade costs in Section 4 below.

In Section 3.1, we totally differentiate the general equilibrium conditions of the model, and derive a linearized system of equations that fully characterizes the transition path of the economy up to first-order. We solve this linearized system in closed form under a wide range of different assumptions about agents' expectations. To illustrate our approach as clearly as possible, we begin in Section 3.2 by considering the simplest case in which agents learn about a one-time unanticipated shock to fundamentals. In Section 3.3, we show that our approach also accommodates the case in which agents learn about any expected convergent sequence of future shocks to fundamentals under perfect foresight. In Section 3.4, we further generalize our analysis to the case in which agents observe an initial shock to fundamentals and form rational expectations about future shocks based on a known stochastic process for fundamentals.

For each specification, we show that the closed-form solution depends on an impact matrix, which captures the initial impact of the shocks to fundamentals on the state variables in the period

⁶In Section F.6 of the online appendix, we provide further details about the numerical algorithm that we use to implement Proposition 2 and solve for the transition path in the non-linear model.

in which they occur, and a transition matrix, which governs the updating of the state variables over time. The impact and transition matrices depend on only the structural parameters of the model and the observed matrices of expenditure shares (\mathbf{S}), income shares (\mathbf{T}), outmigration shares (\mathbf{D}) and immigration shares (\mathbf{E}), and hence provide first-order sufficient statistics for the economy's transition path.

Undertaking an eigendecomposition of the transition matrix, we show that both the rate of convergence to steady-state and the evolution of the state variables along the transition path can be written solely in terms of the eigenvalues and eigenvectors of the transition matrix. We use this lower-dimensional representation to show that the rate of convergence to steady-state is slow and heterogeneous, and the interaction between capital and labor dynamics is central to understanding these persistent and uneven effects of local labor market shocks.

3.1 Transition Path

We suppose that we observe the state variables $\{\ell_t, \mathbf{k}_t\}$ and the trade and migration share matrices $\{\mathbf{S}, \mathbf{T}, \mathbf{D}, \mathbf{E}\}$ of the economy at time $t = 0$. The economy need not be in steady-state at $t = 0$, but we assume that it is on a convergence path towards a steady-state with constant fundamentals $\{\mathbf{z}, \mathbf{b}, \boldsymbol{\kappa}, \boldsymbol{\tau}\}$. We refer to the steady-state implied by these initial fundamentals as the *initial steady-state*. We use a tilde above a variable to denote a log deviation from the initial steady-state ($\tilde{\chi}_{it+1} = \ln \chi_{it+1} - \ln \chi_i^*$) for all variables except for the worker value function, for which with a slight abuse of notation we use the tilde to denote a deviation in levels ($\tilde{v}_{it} \equiv v_{it} - v_i^*$).

We begin by totally differentiating the general equilibrium conditions of the model around the unobserved initial steady-state, holding constant the aggregate labor endowment, trade costs and migration costs. We thus derive the following system of linear equations that fully characterizes the economy's transition path up to first-order:

$$\tilde{\mathbf{p}}_t = \mathbf{S} \left(\tilde{\mathbf{w}}_t - \tilde{\mathbf{z}}_t - (1 - \mu) \left(\tilde{\mathbf{k}}_t - \tilde{\ell}_t \right) \right), \quad (17)$$

$$\begin{aligned} \tilde{\mathbf{k}}_{t+1} &= \tilde{\mathbf{k}}_t + (1 - \beta(1 - \delta)) \left(\tilde{\mathbf{w}}_t - \tilde{\mathbf{p}}_t - \tilde{\mathbf{k}}_t + \tilde{\ell}_t \right) \\ &+ (1 - \beta(1 - \delta)) \frac{1 - \beta}{\beta} (\psi - 1) \mathbb{E}_t \sum_{s=1}^{\infty} \beta^s \left(\tilde{\mathbf{w}}_{t+s} - \tilde{\mathbf{p}}_{t+s} - \tilde{\mathbf{k}}_{t+s} + \tilde{\ell}_{t+s} \right), \end{aligned} \quad (18)$$

$$[\mathbf{I} - \mathbf{T} + \theta(\mathbf{I} - \mathbf{TS})] \tilde{\mathbf{w}}_t = \left[-(\mathbf{I} - \mathbf{T}) \tilde{\ell}_t + \theta(\mathbf{I} - \mathbf{TS}) \left(\tilde{\mathbf{z}}_t + (1 - \mu) \left(\tilde{\mathbf{k}}_t - \tilde{\ell}_t \right) \right) \right], \quad (19)$$

$$\tilde{\ell}_{t+1} = \mathbf{E} \tilde{\ell}_t + \frac{\beta}{\rho} (\mathbf{I} - \mathbf{ED}) \mathbb{E}_t \tilde{\mathbf{v}}_{t+1}, \quad (20)$$

$$\tilde{\mathbf{v}}_t = \tilde{\mathbf{w}}_t - \tilde{\mathbf{p}}_t + \tilde{\mathbf{b}}_t + \beta \mathbf{D} \mathbb{E}_t \tilde{\mathbf{v}}_{t+1}, \quad (21)$$

as shown in Section B.10.4 of the online appendix.

In this system of linear equations, there are no terms in the change in the trade and migration share matrices, because these terms are second-order in the underlying Taylor-series expansion, involving interactions between the changes in productivity and amenities and the resulting changes in trade and migration shares. As we consider small changes in productivity and amenities, these second-order terms converge to zero. Therefore, we are able to write the trade and migration share matrices with no time subscript ($\mathbf{S}, \mathbf{T}, \mathbf{D}, \mathbf{E}$), because they take the same value for small changes in productivity and amenities, whether we use the initial values of the trade and migration share matrices when a fundamental shock occurs, their values in the unobserved initial steady-state, or their values in the new steady-state.

3.2 Transition Dynamics for a One-time Shock

As a first illustration of our approach, we begin by solving for the economy's transition path in response to a one-time shock. We suppose that agents learn at time $t = 0$ about a one-time, unexpected, and permanent change in productivity and amenities from time $t = 1$ onwards. Under this assumption, we can write the sequence of future fundamentals (productivities and amenities) relative to the initial level as $(\tilde{\mathbf{z}}_t, \tilde{\mathbf{b}}_t) = (\tilde{\mathbf{z}}, \tilde{\mathbf{b}})$ for $t \geq 1$, and we can drop the expectation operator in the system of equations (18) through (21).

In Section B.10.4 of the online appendix, we show that the model's transition dynamics can be reduced to the following linear system of second-order difference equations in the state variables:

$$\Psi \tilde{\mathbf{x}}_{t+2} = \Gamma \tilde{\mathbf{x}}_{t+1} + \Theta \tilde{\mathbf{x}}_t + \Pi \tilde{\mathbf{f}}, \quad (22)$$

where $\tilde{\mathbf{x}}_t = \begin{bmatrix} \tilde{\boldsymbol{\ell}}_t \\ \tilde{\mathbf{k}}_t \end{bmatrix}$ is a $2N \times 1$ vector of the state variables; $\tilde{\mathbf{f}} = \begin{bmatrix} \tilde{\mathbf{z}} \\ \tilde{\mathbf{b}} \end{bmatrix}$ is a $2N \times 1$ vector of the shocks to fundamentals; and $\Psi, \Gamma, \Theta,$ and Π are $2N \times 2N$ matrices that depend only on the structural parameters of the model $\{\psi, \theta, \beta, \rho, \mu, \delta\}$ and the observed trade and migration share matrices $\{\mathbf{S}, \mathbf{T}, \mathbf{D}, \mathbf{E}\}$.

We solve this matrix system of equations using the method of undetermined coefficients following Uhlig (1999) to obtain a closed-form solution for the economy's transition path in terms of an impact matrix (\mathbf{R}), which captures the initial impact of the fundamental shocks, and a transition matrix (\mathbf{P}), which governs the evolution of the state variables over time.⁷

Proposition 3. Transition Path. *Suppose that the economy at time $t = 0$ is on a convergence path towards an initial steady-state with constant fundamentals $(\mathbf{z}, \mathbf{b}, \boldsymbol{\kappa}, \boldsymbol{\tau})$. At time $t = 0$, agents*

⁷Relative to the time-series macro literature, our dynamic spatial model features a larger state space of many locations or location-sectors over time. Nevertheless, the use of standard linear algebra techniques allows our approach to accommodate large state spaces, while remaining computationally efficient and easy to implement.

learn about one-time, permanent shocks to productivity and amenities ($\tilde{\mathbf{f}} = \begin{bmatrix} \tilde{\mathbf{z}} \\ \tilde{\mathbf{b}} \end{bmatrix}$) from time $t = 1$ onwards. There exists a $2N \times 2N$ transition matrix (\mathbf{P}) and a $2N \times 2N$ impact matrix (\mathbf{R}) such that the second-order difference equation system in (22) has a closed-form solution of the form:

$$\tilde{\mathbf{x}}_{t+1} = \mathbf{P}\tilde{\mathbf{x}}_t + \mathbf{R}\tilde{\mathbf{f}} \quad \text{for } t \geq 0. \quad (23)$$

Proof. See Section B.10.4 of the online appendix. \square

Both the impact and transition matrices depend on only the structural parameters of the model and the observed trade and migration share matrices (\mathbf{S} , \mathbf{T} , \mathbf{D} , \mathbf{E}), and hence provide first-order sufficient statistics for the dynamics of the spatial distribution of economic activity.

3.2.1 Convergence Dynamics Versus Fundamental Shocks

Using this closed-form solution in Proposition 3, the transition path of the economy's state variables can be additively decomposed into the contributions of convergence dynamics given initial conditions and fundamental shocks. Applying equation (23) across time periods, we obtain:

$$\ln \mathbf{x}_t - \ln \mathbf{x}_{-1} = \underbrace{\sum_{s=0}^t \mathbf{P}^s (\ln \mathbf{x}_0 - \ln \mathbf{x}_{-1})}_{\text{convergence given initial fundamentals}} + \underbrace{\sum_{s=0}^{t-1} \mathbf{P}^s \mathbf{R}\tilde{\mathbf{f}}}_{\text{dynamics from fundamental shocks}} \quad \text{for all } t \geq 1. \quad (24)$$

In the absence of shocks to fundamentals ($\tilde{\mathbf{f}} = \mathbf{0}$), the second term on the right-hand side of equation (24) is zero. In this case, the evolution of the state variables is shaped solely by convergence dynamics given initial conditions, and converges over time to:

$$\ln \mathbf{x}_{\text{initial}}^* = \lim_{t \rightarrow \infty} \ln \mathbf{x}_t = \ln \mathbf{x}_{-1} + (\mathbf{I} - \mathbf{P})^{-1} (\ln \mathbf{x}_0 - \ln \mathbf{x}_{-1}), \quad (25)$$

where $(\mathbf{I} - \mathbf{P})^{-1} = \sum_{s=0}^{\infty} \mathbf{P}^s$ is well-defined under the condition that the spectral radius of \mathbf{P} is smaller than one, a property that we verify empirically.

In contrast, if the economy is initially in a steady-state at time 0, the first term on the right-hand side of equation (24) is zero. In this case, the transition path of the state variables is solely driven by the second term for fundamental shocks, and follows:

$$\tilde{\mathbf{x}}_t = \ln \mathbf{x}_t - \ln \mathbf{x}_0 = \sum_{s=0}^{t-1} \mathbf{P}^s \mathbf{R}\tilde{\mathbf{f}} = (\mathbf{I} - \mathbf{P}^t) (\mathbf{I} - \mathbf{P})^{-1} \mathbf{R}\tilde{\mathbf{f}} \quad \text{for all } t \geq 1. \quad (26)$$

In the period $t = 1$ when the shocks occur, the response of the state variables is $\tilde{\mathbf{x}}_1 = \mathbf{R}\tilde{\mathbf{f}}$. Taking the limit as $t \rightarrow \infty$ in equation (26), the comparative steady-state response is:

$$\lim_{t \rightarrow \infty} \tilde{\mathbf{x}}_t = \ln \mathbf{x}_{\text{new}}^* - \ln \mathbf{x}_{\text{initial}}^* = (\mathbf{I} - \mathbf{P})^{-1} \mathbf{R}\tilde{\mathbf{f}}. \quad (27)$$

A key implication of this additive separability in equation (24) is that we can examine the economy's dynamic response to fundamental shocks separately from its convergence towards an initial steady-state with unchanged fundamentals. Therefore, without loss of generality, we focus in the remainder of this section on an economy that is initially in steady-state.

3.2.2 Spectral Analysis of the Transition Matrix P

We now provide a further analytical characterization of the role of capital and labor dynamics in shaping the economy's gradual adjustment to shocks by undertaking a spectral analysis of the transition matrix. We show that both the speed of convergence to steady-state and the evolution of the state variables along the transition path to steady-state can be written solely in terms of the eigenvalues and eigenvectors of this transition matrix.

Eigendecomposition of the Transition Matrix We begin by undertaking an eigendecomposition of the transition matrix, $P \equiv U\Lambda V$, where Λ is a diagonal matrix of eigenvalues arranged in decreasing order by absolute values, and $V = U^{-1}$. For each eigenvalue λ_h , the h -th column of U (\mathbf{u}_h) and the h -th row of V (\mathbf{v}'_h) are the corresponding right- and left-eigenvectors of P , respectively, such that

$$\lambda_h \mathbf{u}_h = P \mathbf{u}_h, \quad \lambda_h \mathbf{v}'_h = \mathbf{v}'_h P.$$

That is, \mathbf{u}_h (\mathbf{v}'_h) is the vector that, when left-multiplied (right-multiplied) by P , is proportional to itself but scaled by the corresponding eigenvalue λ_h .⁸ We refer to \mathbf{u}_h simply as eigenvectors. Both $\{\mathbf{u}_h\}$ and $\{\mathbf{v}'_h\}$ are bases that span the $2N$ -dimensional vector space.

Eigen-shock We next introduce a particular type of shock to fundamentals that proves useful for characterizing the model's transition dynamics. We define an *eigen-shock* as a shock to productivity and amenities ($\tilde{\mathbf{f}}_{(h)}$) for which the initial impact of these shocks on the state variables ($\mathbf{R}\tilde{\mathbf{f}}_{(h)}$) coincides with a real eigenvector of the transition matrix (\mathbf{u}_h).

In general, there is no necessary reason why any empirical shock should correspond to an eigen-shock. But we now establish that we can characterize the impact of any empirical shock using these eigen-shocks. First, we can solve for these eigen-shocks from the observed data. Assuming that the impact matrix is invertible, which we again verify empirically, we can recover these eigen-shocks from the impact matrix (\mathbf{R}) and the eigenvectors (\mathbf{u}_h) of the transition matrix (P), using $\tilde{\mathbf{f}}_{(h)} = \mathbf{R}^{-1}\mathbf{u}_h$. Both the impact matrix (\mathbf{R}) and the eigenvectors (\mathbf{u}_h) of the transition

⁸Note that P need not be symmetric. This eigendecomposition can be undertaken as long the transition matrix has distinct eigenvalues, a condition that we verify empirically. We construct the right-eigenvectors such that the 2-norm of \mathbf{u}_h is equal to 1 for all h , where note that $\mathbf{v}'_i \mathbf{u}_h = 1$ for $i = h$ and $\mathbf{v}'_i \mathbf{u}_h = 0$ otherwise.

matrix (\mathbf{P}) can be computed using only our observed trade and migration share matrices (\mathbf{S} , \mathbf{T} , \mathbf{D} , \mathbf{E}) and the structural parameters of the model $\{\psi, \theta, \beta, \rho, \mu, \delta\}$.

Second, any empirical productivity and amenity shocks (\mathbf{f}) can be expressed as a linear combination of the eigen-shocks ($\tilde{\mathbf{f}}_{(h)}$), and the weights or loadings in this linear combination can be recovered from a linear projection (regression) of the observed shocks ($\tilde{\mathbf{f}}$) on the eigen-shocks ($\tilde{\mathbf{f}}_{(h)}$). Using this property, the transition path of the state variables in response to any empirical productivity and amenity shocks can be expressed solely in terms of the eigenvalues and eigenvectors of the transition matrix, as summarized in the following proposition.

Proposition 4. Spectral Analysis. *Consider an economy that is initially in steady-state at time $t = 0$ when agents learn about one-time, permanent shocks to productivity and amenities ($\tilde{\mathbf{f}} = \begin{bmatrix} \tilde{\mathbf{z}} \\ \tilde{\mathbf{b}} \end{bmatrix}$) from time $t = 1$ onwards. The transition path of the state variables can be written as a linear combination the eigenvalues (λ_h) and eigenvectors (\mathbf{u}_h) of the transition matrix:*

$$\tilde{\mathbf{x}}_t = \sum_{s=0}^{t-1} \mathbf{P}^s \mathbf{R} \tilde{\mathbf{f}} = \sum_{h=1}^{2N} \frac{1 - \lambda_h^t}{1 - \lambda_h} \mathbf{u}_h \mathbf{v}_h' \mathbf{R} \tilde{\mathbf{f}} = \sum_{h=1}^{2N} \frac{1 - \lambda_h^t}{1 - \lambda_h} \mathbf{u}_h a_h, \quad (28)$$

where the weights in this linear combination (a_h) can be recovered as the coefficients in a linear projection (regression) of the observed shocks ($\tilde{\mathbf{f}}$) on the eigen-shocks ($\tilde{\mathbf{f}}_{(h)}$).

Proof. The proposition follows from the eigendecomposition of the transition matrix: $\mathbf{P} \equiv \mathbf{U} \mathbf{\Lambda} \mathbf{V}$, as shown in Section B.10.7 of the online appendix. \square

Third, the speed of convergence to steady-state for an eigen-shock, as measured by the half-life of convergence to steady-state, depends solely on the associated eigenvalue of the transition matrix, as summarized in the following proposition.

Proposition 5. Speed of Convergence. *Consider an economy that is initially in steady-state at time $t = 0$ when agents learn about one-time, permanent shocks to productivity and amenities ($\tilde{\mathbf{f}} = \begin{bmatrix} \tilde{\mathbf{z}} \\ \tilde{\mathbf{b}} \end{bmatrix}$) from time $t = 1$ onwards. Suppose that these shocks are an eigen-shock ($\tilde{\mathbf{f}}_{(h)}$), for which the initial impact on the state variables at time $t = 1$ coincides with a real eigenvector (\mathbf{u}_h) of the transition matrix (\mathbf{P}): $\mathbf{R} \tilde{\mathbf{f}}_{(h)} = \mathbf{u}_h$. The transition path of the state variables (\mathbf{x}_t) in response to such an eigen-shock ($\tilde{\mathbf{f}}_{(h)}$) is :*

$$\tilde{\mathbf{x}}_t = \sum_{j=1}^{2N} \frac{1 - \lambda_j^t}{1 - \lambda_j} \mathbf{u}_j \mathbf{v}_j' \mathbf{u}_h = \frac{1 - \lambda_h^t}{1 - \lambda_h} \mathbf{u}_h \quad \implies \quad \ln \mathbf{x}_{t+1} - \ln \mathbf{x}_t = \lambda_h^t \mathbf{u}_h,$$

and the half-life of convergence to steady-state is given by:

$$t_i^{(1/2)}(\tilde{\mathbf{f}}) = - \left\lceil \frac{\ln 2}{\ln \lambda_h} \right\rceil,$$

for all state variables $i = 1, \dots, 2N$, where $\tilde{x}_{i\infty} = x_{i,\text{new}}^* - x_{i,\text{initial}}^*$ and $\lceil \cdot \rceil$ is the ceiling function.

Proof. The proposition follows from the eigendecomposition of the transition matrix ($\mathbf{P} \equiv \mathbf{U}\mathbf{\Lambda}\mathbf{V}$), for the case in which the initial impact of the shocks to productivity and amenities on the state variables at time $t = 1$ coincides with a real eigenvector ($\mathbf{R}\tilde{\mathbf{f}}_{(h)} = \mathbf{u}_h$) of the transition matrix (\mathbf{P}), as shown in Section B.10.7 of the online appendix. \square

From Proposition 5, the impact of eigen-shocks ($\tilde{\mathbf{f}}_{(h)}$) on the state variables in each time period is always proportional to the corresponding eigenvector (\mathbf{u}_h), and decays exponentially at a rate determined by the associated eigenvalue (λ_h), as the economy converges to the new steady-state.⁹ These eigenvalues fully summarize the economy’s speed of convergence in response to eigen-shocks, even in our setting with a high-dimensional state space, a rich geography of trade and migration costs, and multiple sources of dynamics from investment and migration.

Each eigen-shock ($\tilde{\mathbf{f}}_{(h)}$) has a different speed of convergence (as captured by the associated eigenvalue λ_h), because this speed of convergence to steady-state depends not only on the structural parameters $\{\psi, \theta, \beta, \rho, \mu, \delta\}$, but also on the incidence of the shock on the labor and capital state variables in each location (as captured by $\mathbf{u}_h = \mathbf{R}\tilde{\mathbf{f}}_{(h)}$). From Proposition 4, any empirical shock ($\tilde{\mathbf{f}}$) can be expressed as a linear combination of the eigen-shocks. Therefore, the speed of convergence will also vary across these empirical shocks, depending on their incidence on the labor and capital state variables in each location.

We use these analytical results in our empirical analysis to characterize the speed of convergence to steady-state and the interaction between the capital and labor adjustment margins in the model. Although our linearization for the economy’s transition path uses techniques from the closed-economy Dynamic Stochastic General Equilibrium (DSGE) literature, there are two key differences in our approach.

First, in the closed-economy macro literature, the convergence properties of the model typically depend only on the eigenvalue with the largest absolute value, because this determines the dynamic properties of the system over long time horizons. In contrast, in our dynamic spatial model, the dynamics of the spatial distribution of economic activity depends on the entire spectrum of eigenvalues and eigenvectors, because the eigen-shocks provide a basis for the space of possible productivity and amenity shocks. Empirical productivity or amenity shocks are typically characterized by non-negligible weights throughout the entire spectrum of eigen-shocks, since these empirical shocks affect many locations simultaneously.

⁹In general, these eigenvectors and eigenvalues can be complex-valued. If the initial impact is the real part of a complex eigenvector \mathbf{u}_h ($\mathbf{R}\tilde{\mathbf{f}} = \text{Re}(\mathbf{u}_h)$), then $\ln \mathbf{x}_{t+1} - \ln \mathbf{x}_t = \text{Re}(\lambda_h^t \mathbf{u}_h) \neq \text{Re}(\lambda_h) \cdot \text{Re}(\lambda_h^{t-1} \mathbf{u}_h)$. That is, the impact no longer decays at a constant rate λ_h . Instead, the complex eigenvalues introduce oscillatory motion as the dynamical system converges to the new steady-state. For expositional purposes, we focus on real-valued eigenvalues and eigenvectors. In our empirical application, the imaginary components of \mathbf{P} ’s eigenvalues are small, implying that oscillatory effects are small relative to the effects that decay exponentially.

Second, closed-economy macro models are concerned with a single aggregate economy, whereas we consider a much higher-dimensional state space, because our framework incorporates many locations or location-sectors.

3.3 Convergent Sequence of Shocks Under Perfect Foresight

We now generalize our analysis of the model's transition dynamics to any convergent sequence of future shocks to productivities and amenities under perfect foresight. In particular, we consider an economy that is somewhere on a convergence path towards an initial steady-state with constant fundamentals at time $t = 0$, when agents learn about a convergent sequence of future shocks to productivity and amenities $\{\tilde{\mathbf{f}}_s\}_{s \geq 1}$ from time $t = 1$ onwards, where $\tilde{\mathbf{f}}_s$ is a vector of log differences in fundamentals between times s and 0 for each location.

Proposition 6. Sequence of Shocks Under Perfect Foresight. *Consider an economy that is somewhere on a convergence path towards steady-state at time $t = 0$, when agents learn about a convergent sequence of future shocks to productivity and amenities $\{\tilde{\mathbf{f}}_s\}_{s \geq 1} = \left\{ \left[\begin{array}{c} \tilde{\mathbf{z}}_s \\ \tilde{\mathbf{b}}_s \end{array} \right] \right\}_{s \geq 1}$ from time $t = 1$ onwards. There exists a $2N \times 2N$ transition matrix (\mathbf{P}) and a $2N \times 2N$ impact matrix (\mathbf{R}) such that the second-order difference equation system in (22) has a closed-form solution of the form:*

$$\tilde{\mathbf{x}}_t = \sum_{s=t+1}^{\infty} (\Psi^{-1}\Gamma - \mathbf{P})^{-(s-t)} \mathbf{R} (\tilde{\mathbf{f}}_s - \tilde{\mathbf{f}}_{s-1}) + \mathbf{R}\tilde{\mathbf{f}}_t + \mathbf{P}\tilde{\mathbf{x}}_{t-1} \quad \text{for all } t \geq 1, \quad (29)$$

with initial condition $\tilde{\mathbf{x}}_0 = \mathbf{0}$ and where Ψ, Γ are matrices from the second-order difference equation (22) and are derived in Section B.10.7 of the online appendix.

Proof. See Section B.10.7 of the online appendix. □

Therefore, even though we consider a general convergent sequence of shocks to productivity and amenities in a setting with many locations connected by a rich geography, and with multiple sources of dynamics from investment and migration, we are again able to obtain a closed-form solution for the transition path of the spatial distribution of economic activity. Both the transition matrix \mathbf{P} and impact matrix \mathbf{R} remain the same as those in the previous subsection, and can be recovered from our observed trade and migration share matrices $\{\mathbf{S}, \mathbf{T}, \mathbf{D}, \mathbf{E}\}$ and the structural parameters of the model $\{\psi, \theta, \beta, \rho, \mu, \delta\}$.

3.4 Stochastic Location Characteristics and Rational Expectations

Most previous research on dynamic spatial models has focused on perfect foresight, because of the challenges of solving non-linear dynamic models in the presence of expectational errors. We

now show that our linearization of the economy's transition path also accommodates the case in which agents observe an initial shock to fundamentals and form rational expectations about future shocks based on a known stochastic process for fundamentals.

In particular, we assume the following AR(1) process for fundamentals, which allows shocks to productivity and amenities to have permanent effects on the level of these variables, and hence to affect the steady-state equilibrium:

$$\begin{aligned}\Delta \ln z_{it+1} &= \rho^z \Delta \ln z_{it} + \varpi_{it}^z, & |\rho^z| < 1, \\ \Delta \ln b_{it-1} &= \rho^b \Delta \ln b_{it} + \varpi_{it}^b, & |\rho^b| < 1,\end{aligned}\tag{30}$$

where we use $\Delta \ln$ to denote log changes between two periods, such that $\Delta \ln z_{it} \equiv \ln z_{it} - \ln z_{it-1}$; $\rho^z = \rho^b = 0$ corresponds to the special case of a random walk; and ϖ_{it}^z and ϖ_{it}^b are mean zero and independently and identically distributed innovations. Given this assumed AR(1) process, we can write the expected values of these future fundamental shocks as:

$$\mathbb{E}_t [\Delta \ln \mathbf{f}_{t+s}] = \mathbf{N}^s \Delta \ln \mathbf{f}_t, \quad \mathbf{N} \equiv \begin{bmatrix} \rho^z \cdot \mathbf{I}_{N \times N} & \mathbf{0}_{N \times N} \\ \mathbf{0}_{N \times N} & \rho^b \cdot \mathbf{I}_{N \times N} \end{bmatrix}, \tag{31}$$

where $\mathbb{E}_t [\cdot]$ is the expectation conditional on the realizations of shocks up to time t .

Proposition 7. Stochastic Fundamentals and Rational Expectations. *Suppose that productivity and amenities evolve stochastically according to the AR(1) process (30) and agents have rational expectations. There exists a $2N \times 2N$ transition matrix (\mathbf{P}) and a $2N \times 2N$ impact matrix (\mathbf{R}) such that the evolution of the economy's state variables (\mathbf{x}_t) has the following closed-form solution:*

$$\Delta \ln \mathbf{x}_{t+1} = \mathbf{P} \Delta \ln \mathbf{x}_t + \mathbf{R} \Delta \ln \mathbf{f}_t + \sum_{s=0}^{\infty} (\Psi^{-1} \Gamma - \mathbf{P})^{-s} \mathbf{R} \mathbf{N}^{s+1} (\Delta \ln \mathbf{f}_t - \Delta \ln \mathbf{f}_{t-1}). \tag{32}$$

Proof. See Section B.10.7 of the online appendix. □

In this case, the innovations in fundamental shocks at time t not only affect the current-period state variables (ℓ_t, \mathbf{k}_t), but also affect the entire expected sequence of future fundamental shocks, because of serial correlation in fundamental shocks (ρ^z and ρ^b not equal to zero).

4 Extensions

The tractability of our dynamic spatial model allows for a large number of extensions and generalizations. In Subsection 4.1, we incorporate shocks to trade and migration costs. In Subsection 4.2, we allow productivity and amenities to have an endogenous component that reflects agglomeration forces, as well as an exogenous component of fundamentals. In Subsection 4.3, we discuss extensions with multiple final goods sectors and input-output linkages. In Subsection 4.4, we discuss a number of additional extensions, including trade deficits, residential capital (housing), and investments in other locations.

4.1 Shocks to Trade and Migration Costs

Our generalization of dynamic exact-hat algebra methods in Proposition 2 naturally accommodates shocks to trade and migration costs, as well as to productivity and amenities. Our spectral analysis of the economy's transition path also continues to apply once we allow for these bilateral shocks to trade and migration costs. To preserve the same matrix representation of the economy's transition path, we define measures of outgoing and incoming changes in trade and migration costs, which are weighted averages of the bilateral changes in trade and migration costs. In Section D.1 of the online appendix, we report the generalization of the matrix system in equations (18)-(21) above to incorporate these outgoing and incoming shocks to trade and migration costs.

4.2 Agglomeration Forces

Our dynamic spatial model also generalizes to admit agglomeration forces, such that productivity and amenities have both exogenous and endogenous components. The exogenous components capture locational fundamentals, such as climate and access to natural water: \bar{z}_{it} and \bar{b}_{it} . We follow the standard approach of modelling agglomeration forces as constant elasticity functions of own location population: $z_{it} = \bar{z}_{it}\ell_{it}^{\eta^z}$ and $b_{it} = \bar{b}_{it}\ell_{it}^{\eta^b}$, where $\eta^z > 0$ and $\eta^b > 0$ parameterize agglomeration forces for productivity and amenities, respectively.¹⁰

Introducing agglomeration forces magnifies the impact of exogenous differences in fundamentals on the spatial distribution of economic activity. Furthermore, depending on the strength of these agglomeration forces, there can either be a unique steady-state equilibrium or multiple steady-state equilibria for the spatial distribution of economic activity. Again we obtain an analytical characterization of the conditions for the existence and uniqueness of the steady-state equilibrium, as summarized in the following proposition.

Proposition 8. (A) *There exists a unique steady-state spatial distribution of economic activity $\{\ell_i^*, k_i^*\}$ given the exogenous fundamentals $\{\bar{z}_i, \bar{b}_i, \tau_{ni}, \kappa_{ni}\}$ if the following condition holds:*

$$\max \left\{ \left| 1 - \frac{\beta\eta^b}{\rho} \right|, 1 + \mu\theta \right\} < \left| \frac{\beta}{\rho} (1 - \eta^b - \eta^z\theta - \eta^b\mu\theta) + (1 + \mu\theta) \right|. \quad (33)$$

(B) *A sufficient condition for (33) to hold is that agglomeration forces are small ($\eta^b + \eta^z\theta + \eta^b\mu\theta < 1$).*

Proof. See Section D.2 of the web appendix. □

¹⁰Although for simplicity we assume that agglomeration and dispersion forces only depend on a location's own population, our framework can be further generalized to incorporate spillovers across locations, as in Ahlfeldt et al. (2015) and Allen et al. (2020). While we focus on agglomeration forces ($\eta^z > 0$ and $\eta^b > 0$), it is straightforward to also allow for additional dispersion forces ($\eta^z < 0$ and $\eta^b < 0$).

From part (A) of Proposition 8, whether there is a unique steady-state depends on the strength of agglomeration forces (η^z, η^b), trade elasticity (θ), capital intensity (μ), the discount rate (β) and the dispersion of idiosyncratic preferences (ρ). From part (B) of Proposition 8, a sufficient condition for a unique steady-state is that the agglomeration forces (η^z, η^b) are sufficiently small. As these parameters converge towards zero, we obtain our baseline specification without agglomeration forces, in which there is necessarily a unique steady-state equilibrium, as shown in Section 2.6 above. Additionally, an increase in the dispersion of idiosyncratic preferences (a larger value for ρ) necessarily increases the range of values for the other parameters for which a unique steady-state equilibrium exists, as shown in the proof of the proposition.

4.3 Multi-sectors and Input-Output Linkages

Our dynamic spatial model also can be extended to incorporate multiple sectors and input-output linkages. In our multi-sector generalizations, we consider two different assumptions about capital mobility across sectors. In Section D.3 of the online appendix, we assume that installed capital is specific to a location, but mobile across sectors within locations. In Section D.4 of the online appendix, we assume installed capital is specific to both a location and sector. This second specification corresponds to a dynamic spatial version of the traditional specific-factors model, in which there are migration frictions for the mobile factor across locations and sectors, and there is endogenous accumulation of the factor specific to each location and sector over time. In both cases, we assume Cobb-Douglas preferences across sectors and CES preferences across the goods supplied by locations within sectors, as in Costinot et al. (2012).

In Section D.5 of the online appendix, we further generalize these multi-sector specifications to allow for input-output linkages, where the production technology in each sector now uses the two primary factors of labor and capital together with intermediate inputs according to a Cobb-Douglas functional form, as in Caliendo and Parro (2015).

In each case, consumption, production, trade and migration are modelled in a similar way as in our baseline specification. The set of equilibrium conditions for the value function (14), population flow (15), and goods market clearing (12) extends naturally from describing each location to describing each location-sector, and the capital market clearing and accumulation condition (11) generalizes in a form that depends on the capital mobility assumption.

Both our generalization of dynamic exact-hat algebra methods to incorporate forward-looking investment and our spectral analysis of the economy's transition path extend to these settings with multiple sectors and input-output linkages.

4.4 Other Extensions

Our approach also accommodates a number of other extensions. In our baseline specification, we model trade between locations as in [Armington \(1969\)](#), in which goods are differentiated by location of origin. In [Section C](#) of the online appendix, we establish a number of isomorphisms, in which we show that our results hold throughout the class of trade models with a constant trade elasticity. In [Section D.6](#) of the online appendix, we incorporate trade deficits following the conventional approach of the quantitative international trade literature of treating these deficits as exogenous. In [Section D.7](#) of the online appendix, we allow capital to be used residentially (for housing) as well as commercially (in production). In [Section D.8](#) of the online appendix, we allow landlords to invest in other locations subject to bilateral investment costs and idiosyncratic heterogeneity in the productivity of these investments. In each case, our generalization of dynamic exact-hat algebra methods and our spectral analysis continue to apply.

5 Quantitative Analysis

We now use our dynamic spatial model to understand the observed decline in rates of convergence in income per capita across U.S. states since the early 1960s. Our framework is particularly well suited for this application, because it allows for roles of initial conditions and shocks to fundamentals, and both capital and labor dynamics shape the rate of convergence to steady-state. Our framework also captures key features of the observed data, such as gravity equation relationships for bilateral flows of goods and migrants between states.

In our baseline specification, we consider a version of our single-sector model, augmented to take account of the empirically-relevant distinction between traded and non-traded goods.¹¹ We also implement our multi-sector extension with location-sector-specific capital from [Section 4.3](#) for the shorter time period from 1999-2015 for which the sector-level data are available.

In [Subsection 5.1](#), we discuss our data sources and the parameterization of the model. In [Subsection 5.2](#), we provide evidence of a decline in rates of convergence in income per capita across U.S. states since the early 1960s. In [Subsection 5.3](#), we examine the extent to which this observed decline in income convergence is explained by initial conditions versus fundamental shocks, and quantify the respective contributions of capital and labor dynamics.

In [Subsection 5.4](#), we use our spectral analysis to provide evidence on the speed of convergence to steady-state and the role of the interaction between capital and labor dynamics in shaping the persistent and heterogeneous impact of local labor market shocks. In [Subsection 5.5](#), we demonstrate that we find a similar pattern of results for our multi-sector extension.

¹¹Therefore, our single-sector model in our empirical implementation features a single traded sector and a single non-traded sector, as developed in detail in [Section E](#) of the online appendix.

5.1 Data and Parameterization

Our main source of data for our baseline quantitative analysis from 1965-2015 is the national economic accounts of the Bureau of Economic Analysis (BEA), which report population, gross domestic product (GDP) and the capital stock for each U.S. state.¹² We focus on the 48 contiguous U.S. states plus the District of Columbia, excluding Alaska and Hawaii, because they only became U.S. states in 1959 close to the beginning of our sample period, and could be affected by idiosyncratic factors as a result of their geographical separation. We deflate GDP and the capital stock to express them in constant (2012) prices.

We construct bilateral five-year migration flows between U.S. states from the U.S. population census from 1960-2000 and from the American Community Survey (ACS) after 2000. We define a period in the model as equal to five years to match these observed data. We interpolate between census decades to obtain five-year migration flows for each year of our sample period. To take account of international migration to each state and fertility/mortality differences across states, we adjust these migration flows by a scalar for each origin and destination state, such that origin population in year t pre-multiplied by the migration matrix equals destination population in year $t + 1$, as required for internal consistency.

We construct the value of bilateral shipments between U.S. states from the Commodity Flow Survey (CFS) from 1993-2017 and its predecessor the Commodity Transportation Survey (CTS) for 1977. We again interpolate between reporting years and extrapolate the data backwards in time before 1977 using relative changes in the income of origin and destination states, as discussed in further detail in Section G of the online appendix. For our baseline quantitative analysis with a single traded and non-traded sector, we abstract from direct shipments to and from foreign countries, because of the relatively low level of U.S. trade openness, particularly towards the beginning of our sample period.

For our multi-sector extension from 1999-2015, we construct data for the 48 contiguous U.S. states, 43 foreign countries and 23 economic sectors, yielding a total of 2,093 region-sector combinations, where a region is either a U.S. state or a foreign country. We allow for trade across all region-sectors, and for migration across all U.S. states and sectors. We obtain sector-level data on value added, employment and the capital stock for each U.S. state from the national economic accounts of the Bureau of Economic Analysis (BEA).

We construct migration flows between U.S. states in each sector by combining data from the U.S. population census, American Community Survey (ACS), and Current Population Survey (CPS), as discussed in Section G of the online appendix. We use the value of bilateral shipments between U.S. states in each sector from the Commodity Flow Survey (CFS), interpolating between

¹²For further details on the data sources and definitions, see the data appendix in Section G of the online appendix.

reporting years. We measure foreign trade for each U.S. state and sector using the data on foreign exports by origin of movement (OM) and foreign imports by state of destination (SD) from the U.S. Census Bureau.¹³ For each foreign country and sector, we obtain data on value added, employment and the capital stock from the World Input-Output Tables (WIOT).

To focus on the impact of introducing forward-looking investment decisions, our baseline specification assumes standard values of the model’s structural parameters from the existing empirical literature. We assume a trade elasticity of $\theta = 5$, as in [Costinot and Rodríguez-Clare \(2014\)](#). We set the 5-year discount rate equal to the conventional value of $\beta = (0.95)^5$. We assume an elasticity of intertemporal substitution of $\psi = 1$, which corresponds to logarithmic intertemporal utility. We assume a value for the migration elasticity of $\rho = 3\beta$, which is in line with the values in [Bryan and Morten \(2019\)](#), [Caliendo et al. \(2019\)](#) and [Fajgelbaum et al. \(2019\)](#). We set the share of labor in value added to $\mu = 0.65$, as a central value in the macro literature. We assume a five percent annual depreciation rate, such that the 5-year depreciation rate is $\delta = 1 - (0.95)^5$, which is again a conventional value in the macro and productivity literatures. We later report comparative statics for how changes in each these model parameters affect the speed of convergence to steady-state using our closed-form solutions for the economy’s transition path.

5.2 Income Convergence

We begin by establishing a substantial decline in the rate of convergence in income per capita across U.S. states in recent decades, before using our theoretical framework to understand the reasons for this decline in income convergence.

In [Figure 1](#), we display the annualized rate of growth of income per capita against its initial level for each U.S. state for different sub-periods, where the size of the circles is proportional to initial state employment. We also show the regression relationship between these variables as the red dotted line, where the estimated slope coefficient is sometimes referred to as a measure of β -convergence. In the opening sub-period from 1963-80 ([Panel \(a\)](#)), we find substantial income convergence, with a negative and statistically significant coefficient of -0.0236 (standard error 0.0038), and a regression R-Squared of 0.476. This estimated coefficient is close to the 0.02 estimated by [Barro and Sala-i-Martin \(1992\)](#) for the longer time period from 1880-1988. By the middle sub-period from 1980-2000, we find that this relationship substantially weakens, with the slope coefficient falling by nearly one half to -0.0135 (standard error 0.0039), and a smaller regression R-squared of 0.209. By the closing sub-period from 2000-2017, we find income divergence rather

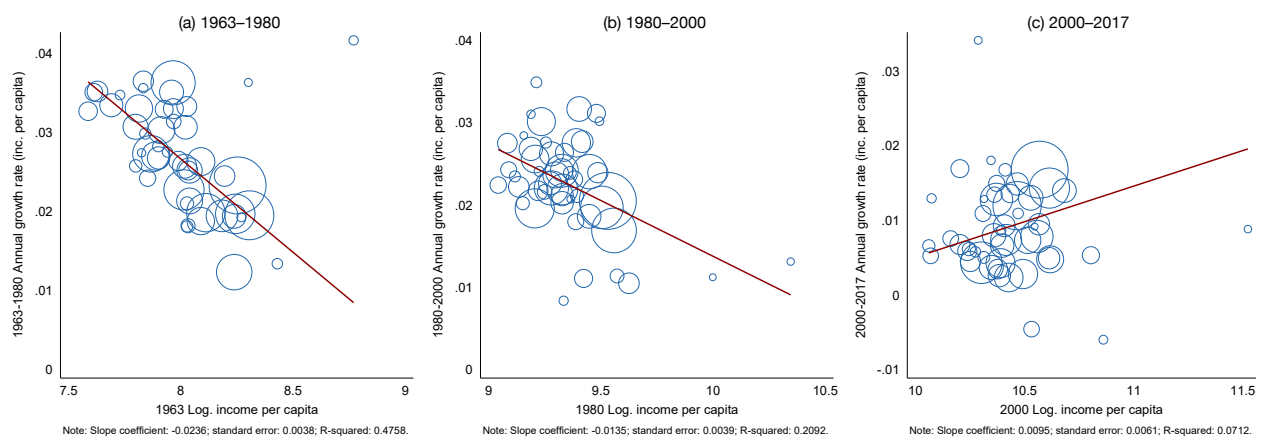
¹³The Census Bureau constructs these data from U.S. customs transactions, aiming to measure the origin of the movement of each export shipment and the destination of each import shipment. Therefore, these data differ from measures of exports and imports constructed from port of exit/entry, and from the data on the exports of manufacturing enterprises (EME) from the Annual Survey of Manufactures (ASM). See <https://www.census.gov/foreign-trade/aip/elom.html> and [Cassey \(2009\)](#).

than income convergence, with a positive but not statistically significant coefficient of 0.0095 (standard error 0.0061), and a regression R-squared of 0.071.

Our framework is well suited for understanding the reasons for this declining income convergence. As in closed-economy growth models, we incorporate capital accumulation as a key mechanism for income convergence. In Section F.1 of the online appendix, we show that U.S. states differ substantially in terms of the evolution of their capital-labor ratios over our sample period, highlighting the empirical relevance of this capital accumulation mechanism.

Unlike these closed-economy growth models, we allow for bilateral migration between states, which is also a central feature of the observed data. In Subsection F.2, of the online appendix, we provide evidence of substantial gross and net migration flows between U.S. states. These net migration flows directly affect state population shares, and indirectly affect state capital stocks, through the incentives for capital accumulation.

Figure 1: Growth and Initial Level of Income Per Capita



Notes: Vertical axis shows the annualized rate of growth of income per capita for the relevant sub-period; horizontal axis displays the initial level of income per capita at the beginning of the relevant sub-period; circles correspond to U.S. states; the size of each circle is proportional to state employment; the dashed red line shows the linear regression relationship between the two variables.

Finally, our framework also incorporates bilateral trade between states as an important determinant of the consumption price index in each U.S. state. This consumption price index in turn influences incentives to migrate, through the real wage in terms of the consumption price index (w_i/p_i), and incentives to accumulate capital, through the real rental rate in terms of the consumption price index (r_i/p_i). In Subsection F.3 of the online appendix, we show that the model’s gravity equation predictions provide a good approximation to the observed data on both bilateral goods and migration flows between U.S. states.

5.3 Convergence to Steady-State Versus Fundamental Shocks

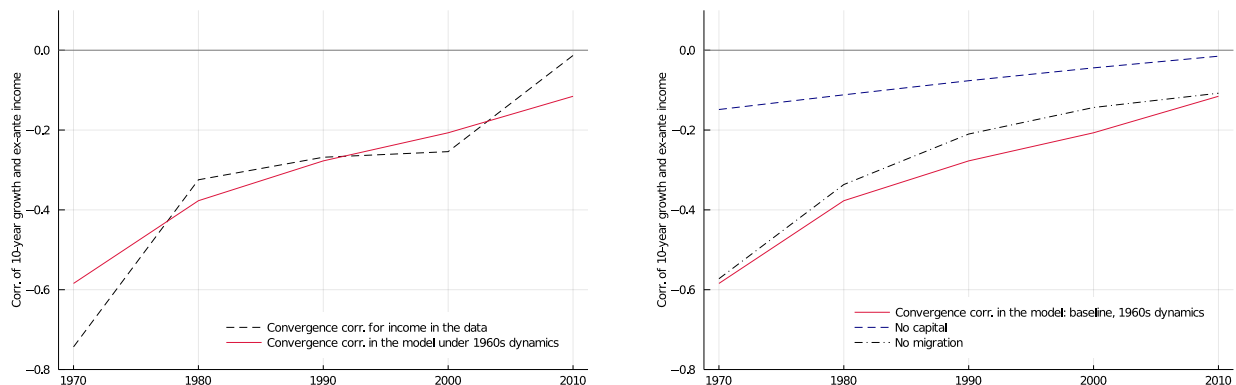
Within our framework, the observed decline in regional income convergence can be explained by either initial conditions (gradual convergence to steady-state) or shocks to fundamentals (productivity, amenities, trade costs and migration frictions). In each case, the dynamic adjustment of the economy depends on both capital accumulation and migration, and we provide evidence on the role of these two sources of dynamics and their interaction with one another.

Initial Conditions Versus Fundamental Shocks To examine the role of initial conditions, we use our generalization of dynamic exact-hat algebra in Proposition 2. Starting from the observed equilibrium in the data at the beginning of our sample period, we solve for the economy's transition path to steady-state in the absence of any further changes in fundamentals. We thus obtain counterfactual values for income per capita in each year implied by initial conditions alone.

Figure 2: Initial Conditions and Income Convergence

(a) Actual and Counterfactual Convergence

(b) Counterfactual Convergence (With and Without Investment and Migration)



Note: Correlation coefficients between the 10-year ahead log growth in income per capita and its initial level in each year from 1970-2010; in the left panel, the dashed black line show these correlations in the data; in both panels, red solid line shows the correlation coefficients for counterfactual income per capita, based on starting at the observed equilibrium in the data at the beginning of our sample period, and solving for the economy's transition path to steady-state in the absence of any further shocks to fundamentals; in the right panel, the black dashed line shows results for the special case with no capital accumulation, and the black dashed-dotted line shows results for the special case with no migration.

For both the actual and counterfactual values of income per capita, we correlate the 10-year ahead log growth in income per capita with its initial level in each year from 1970-2010. In Figure 2a, we display these correlation coefficients over time, which summarize the strength of regional convergence for actual income per capita (dashed black line) and counterfactual income per capita in the absence of any further fundamental shocks (solid red line). We find that the decline in the rate of regional convergence is around the same magnitude for both counterfactual and actual

income per capita. Therefore, these results provide a first key piece of evidence that much of the observed decline in the rate of income convergence is explained by initial conditions at the beginning of our sample period rather than by any subsequent fundamental shocks. Additionally, the fact that it takes decades for the decline in both actual and counterfactual income convergence to occur is indicative of slow convergence to steady-state.

Capital Accumulation and Migration Dynamics To examine the respective roles played by capital accumulation and migration dynamics in this effect of initial conditions, we use our generalization of dynamic exact-hat algebra from Proposition 2 for the special cases of the model with no investment (in which case our framework reduces to the dynamic discrete choice migration model of [Caliendo et al. 2019](#)) and no migration (in which case the population share of each state is exogenous at its 1965 level). Again we start at the observed equilibrium in the data at the beginning of our sample period, and we solve for the transition to steady-state in the absence of any further changes to fundamentals for these two special cases of the model. We thus obtain counterfactual values for income per capita in each year implied by initial conditions alone for these special cases of the model with no investment and no migration.

Using these counterfactual predictions, we again correlate the 10-year ahead log growth in income per capita with its initial level from 1970-2010. In Figure 2b, we display these correlation coefficients over time for the full model (replicating the results from Figure 2a, as shown by the solid red line), the model with no investment (dashed line), and the model with no migration (dotted-dashed line). We find substantial contributions to the observed decline in income convergence over time from both investment and migration dynamics. Capital accumulation is more important than migration for these dynamics of income per capita, highlighting the relevance of incorporating investment decisions into dynamic spatial models. Nevertheless, even in the model with no capital, we find a decline in the correlation coefficient for income convergence of around 20 percentage points. Furthermore, migration dynamics are naturally central to matching the observed changes in the population distribution across U.S. states over time.

Steady-State Gaps in Population Shares These findings that initial conditions explain much of the observed decline in income convergence over time suggests that U.S. states were substantially away from steady-state at the beginning of our sample period. We now provide further evidence on the evolution of these steady-state gaps over the course of our sample period. First, we compute the implied steady-state population share for each U.S. state and year, by using our generalization of exact-hat algebra in Proposition 2 to solve for the economy's transition path in the absence of further changes in fundamentals, given the observed values of the labor and capital state variables in each year. Second, we calculate the implied steady-state gap in population

shares for each year separately, measured as the log-ratio of the actual population share to the steady-state population share.

In Figure 3, we display the evolution of these steady-state gaps for each U.S. state using the solid gray lines. We also show the population-share weighted average of these steady-state gaps using the black dashed lines for four broad geographical regions: The Rust Belt; the Sun Belt; Other Northern States; and Other Southern States. We use standard definitions of these four regions. Following [Alder et al. \(2019\)](#), we define the Rust Belt as the states of Illinois, Indiana, Michigan, New York, Ohio, Pennsylvania, West Virginia and Wisconsin, and the Sun Belt as the states of Arizona, California, Florida, New Mexico and Nevada. Other Southern States include all former members of the Confederacy, except those in the Sun Belt. Other Northern States comprise all the Union states from the U.S. Civil War, except those in the Rust Belt or Sun Belt.¹⁴

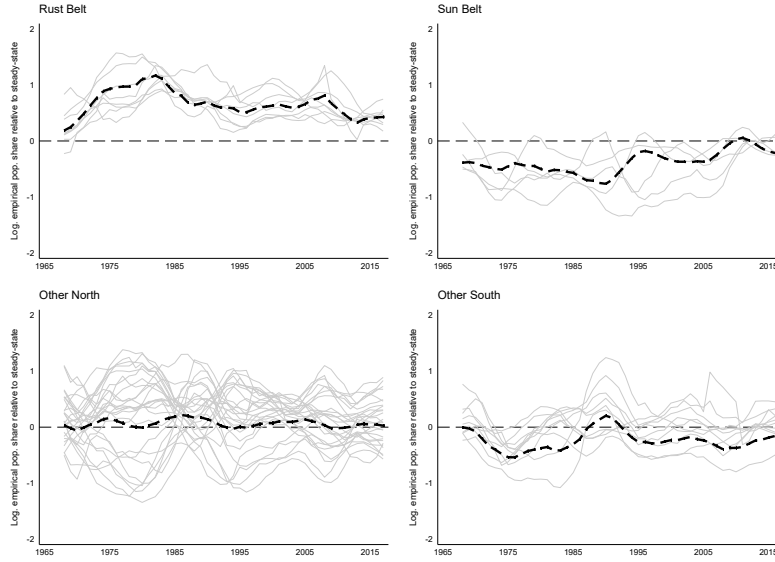
Four key features are apparent. First, consistent with our results above, the population shares of the Rust Belt states were substantially above steady-state, and the population shares of the Sun Belt states were substantially below steady-state, even at the beginning of our sample period. Second, Rust Belt states move substantially further away from steady-state from the mid-1960s until around 1980, which in the model is driven by shocks to fundamentals, such as productivity and amenities. Third, both individual states and these four geographical regions remain persistently away from steady-state for decades, providing further evidence of slow convergence towards steady-state. Fourth, individual states are typically closer to steady-state towards the end of our sample period than in previous decades, which contributes towards an observed decline in rates of inter-state migration in the second half of our sample period.¹⁵

Predicted Population Growth Based on Initial Conditions In Section F.4 of the online appendix, we provide further evidence on the explanatory power of initial conditions, by regressing actual population growth on predicted population growth based on convergence towards an initial steady-state with unchanged fundamentals. Importantly, predicted population growth is calculated using only the initial values of the labor and capital state variables and the initial trade and migration share matrices, and uses no information about subsequent population growth.

¹⁴Therefore, “Other Southern” includes Alabama, Arkansas, Georgia, Louisiana, Mississippi, North Carolina, South Carolina, Tennessee, Texas, and Virginia. “Other Northern” includes Colorado, Connecticut, Delaware, District of Columbia, Idaho, Iowa, Kansas, Kentucky, Maine, Maryland, Massachusetts, Minnesota, Missouri, Montana, Nebraska, New Hampshire, New Jersey, North Dakota, Oklahoma, Oregon, Rhode Island, South Dakota, Utah, Vermont, Washington and Wyoming.

¹⁵This decline in rates of inter-state migration is in line with the findings of [Kaplan and Schulhofer-Wohl \(2017\)](#) and [Molloy et al. \(2011\)](#). Consistent with [Hyatt et al. \(2018\)](#), we find that this decline is smaller in the population census data than in Current Population Survey (CPS) data.

Figure 3: Steady-State Gaps of Population Shares for U.S. States over Time



Note: Gray lines show gaps of population shares from steady-state for each U.S. state and year; black dashed lines show the population-weighted average of these steady-state gaps for the four geographical regions of the Rust Belt, Sun Belt, Other Northern States and Other Southern States, as defined in the main text; we compute the steady-state population share for each U.S. state and year by using our generalization of exact-hat algebra in Proposition 2 to solve for the economy’s transition path in the absence of further changes in fundamentals, given the observed values of the labor and capital state variables in each year; gap from steady-state is measured as the log-ratio of actual population share to steady-state population share.

We estimate these regressions for each of the periods 1965-2015, 1975-2015, 1985-2015 and 1995-2015, as reported in Section F.4 of the online appendix. Even for the period from 1965-2015, for which there is the greatest residual contribution from shocks to fundamentals, we find a strong positive and statistically significant relationship between actual and predicted population growth, with a regression slope (standard error) of 0.64 (0.18) and R-squared of 0.19. For all periods starting in the mid-1970s onwards, we find an even stronger relationship between actual and predicted population growth. For example, for the period from 1975-2015, we estimate a regression slope (standard error) of 0.99 (0.095) and R-squared of 0.82.

We show that this explanatory power of predicted population growth is not driven by mean reversion. We find that controlling for initial log population and the initial log capital stock, as well as initial log population growth, has little impact on the estimated coefficient on predicted population growth or the regression R-squared.

5.4 Spectral Analysis

We now use our spectral analysis to quantify the role of capital and labor dynamics in shaping the speed of convergence to steady-state. In bringing together these two adjustment margins of

capital accumulation and migration, we show that they interact systematically with one another in the model to shape the persistence and heterogeneity in the impact of local labor market shocks. We use our eigendecomposition of the transition matrix (P) in Propositions 4 and 5 above to analytically characterize this interaction.

Speed of Convergence to Steady-State We begin by using Proposition 5 to compute half-lives of convergence to steady-state. In particular, we compute the number of years for the state variables to converge half of the way towards steady-state for eigen-shocks ($\tilde{f}_{(h)}$), for which the initial impact of these shocks on the state variables ($R\tilde{f}$) corresponds to a real eigenvector of the transition matrix (u_h). We have as many half-lives as eigen-shocks, where each of these eigen-shocks corresponds to a different incidence of shocks to productivity and amenities across the labor and capital state variables in each location.

We find that the speed of convergence to steady-state is typically slow, with an average half life across the eigen-shocks of around 20 years. Therefore, our theoretical framework provides microeconomic foundations for reduced-form empirical findings of persistent impacts of local labor market shocks, as found for example for the China shock in the United States in Autor et al. (2013) and Brazil’s trade liberalization in Dix-Carneiro and Kovak (2017). We also find substantial variation in the speed of convergence to steady-state across the eigen-shocks, with half lives ranging from less than 10 years to more than 70 years, depending on the incidence of these shocks across the state variables. Hence, our theoretical framework also rationalizes heterogeneous effects of local labor market shocks, as found in Eriksson et al. (2019).

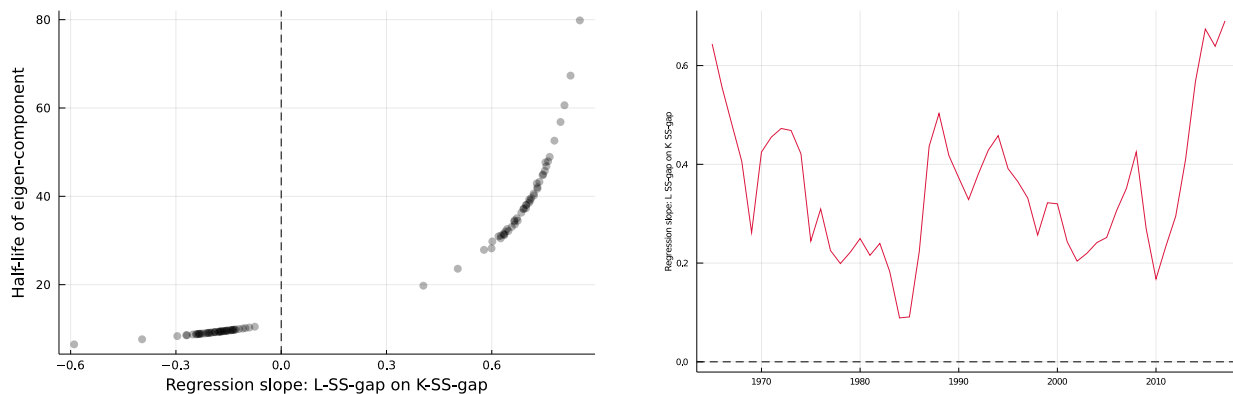
Speed of Convergence and the Labor and Capital State Variables We now characterize the role of the interaction between the capital and labor adjustment margins in determining the speed of convergence towards steady-state. In the left panel of Figure 4, we use the vertical axis to display the half-life of convergence to steady-state for each eigen-shock, as determined by the associated eigenvalue (λ_h). On the horizontal axis, we show the slope coefficients from regressions across locations of the labor gap from steady-state ($\tilde{\ell}_h$) on the capital gap from steady-state (\tilde{k}_h) for each eigen-shock, as reflected in the eigenvector summarizing the initial impact of the shock on the state variables ($u_h = R\tilde{f}_{(h)}$). We display results for the year 2000, but find the same pattern for each year of our sample period. We find a strong, positive relationship between the half-life of convergence to steady-state and the correlation between the gaps from steady-state for the two state variables. We observe low half-lives (fast convergence) for negative correlations and high half-lives (slow convergence) for positive correlations.

This interaction between the capital and labor adjustment margins arises through the marginal products of capital and labor in the production technology. A high stock of capital in a location

Figure 4: Half-lives of Convergence to Steady-State and the Regression Slope Coefficient Between the Labor and Capital Gaps from Steady-State

(a) Half-Lives and the Regression Slope Coefficient Between the Labor and Capital Gaps from Steady-State ($\mathbf{u}_h = \mathbf{R}\tilde{\mathbf{f}}_{(h)}$) for each Eigen-shock in 2000

(b) Regression Slope Coefficient Between the Empirical Labor and Capital Gaps from Steady-State ($\tilde{\ell}_t, \tilde{\mathbf{k}}_t$) over Time



Note: In the left panel, half-life on the vertical axis corresponds to the time in years for the state variables to converge half of the way towards steady-state for an eigen-shock in 2000, for which the initial impact of the shock to productivity and amenities on the state variables ($\mathbf{R}\tilde{\mathbf{f}}_{(h)}$) corresponds to an eigenvector (\mathbf{u}_h) of the transition matrix (\mathbf{P}), as determined by the corresponding eigenvalue (λ_h); the horizontal axis shows slope coefficients from regressions across locations of the labor gap ($\tilde{\ell}_h$) from steady-state on the capital gap ($\tilde{\mathbf{k}}_h$) from steady-state for each eigen-shock in 2000 ($\mathbf{u}_h = \mathbf{R}\tilde{\mathbf{f}}_{(h)}$); each dot corresponds to a different eigencomponent in 2000; In the right panel, the vertical axis shows the regression slope coefficient of the empirical labor gap ($\tilde{\ell}_t$) from steady-state on the empirical capital gap ($\tilde{\mathbf{k}}_t$) from steady-state in each year.

raises the marginal productivity of labor, while a high population in a location raises the marginal productivity of capital. Therefore, if both capital and labor are above steady-state in a location, this slows convergence, because the high capital stock retards the downward adjustment of labor, and the high population dampens the downward adjustment of capital. Similarly, if both capital and labor are below steady-state, this also slows convergence, by an analogous line of reasoning. In contrast, if capital is above steady-state whereas labor is below steady-state, this speeds up convergence, because the high capital stock attracts labor, and the low population discourages capital accumulation.

Since empirical productivity and amenity shocks are weighted averages of these eigen-shocks, the speed of convergence to steady-state will vary across these empirical shocks, depending on their incidence on the labor and capital state variables, as reflected in their weights or loadings on the individual eigen-shocks. In the right panel of Figure 4, we display the slope coefficients from regressions of the empirical labor gap from steady-state ($\tilde{\ell}_t$) on the empirical capital gap from steady-state ($\tilde{\mathbf{k}}_t$) for each year of our sample period, as computed using our generalization of dynamic exact-hat algebra in Proposition 2 above. We find that these empirical gaps from steady-state are positively correlated for each year of our sample period, which rationalizes our findings

above of slow convergence to steady-state and hence an important role for initial conditions.

Speed of Convergence and Productivity and Amenity Shocks In Figure 4, we examined the relationship between the speed of convergence and the initial impact of the eigen-shock on the state variables (as captured by the eigenvector $\mathbf{u}_h = \mathbf{R}\tilde{\mathbf{f}}_{(h)}$). In contrast, we now turn to consider the corresponding relationship between the speed of convergence and the pattern of productivity and amenity shocks captured by each eigen-shock (using $\tilde{\mathbf{f}}_{(h)} = \mathbf{R}^{-1}\mathbf{u}_h$). In the left panel of Figure 5, we again use the vertical axis to display the half-life of convergence to steady-state for each eigen-shock, as determined by the associated eigenvalue (λ_h). On the horizontal axis, we show the slope coefficients from regressions across locations of the productivity shocks ($\tilde{\mathbf{z}}_{(h)}$) on the amenity shocks ($\tilde{\mathbf{b}}_{(h)}$) for each eigen-shock (using $\tilde{\mathbf{f}}_{(h)} = \mathbf{R}^{-1}\mathbf{u}_h$). Again we display results for the year 2000, but find the same pattern of results for each year of our sample period. We find a strong, positive relationship between the half-life of convergence to steady-state and the correlation between the productivity and amenity shocks, with low half-lives (fast convergence) for negative correlations, and high half-lives (slow convergence) for positive correlations.

This pattern of results again highlights the interaction between the capital and labor adjustment margins in the model. On the one hand, a positive productivity shock directly raises the marginal productivity of both capital and labor in the production technology, which raises the new steady-state values of labor and capital relative to their initial values. On the other hand, a positive amenity shock directly raises the expected value of living in a location, which increases the new steady-state value of labor relative to its initial value. Therefore, a positive correlation between these two fundamental shocks induces a positive correlation between the labor and capital steady-state gaps, which in turn implies slow convergence to steady-state, as discussed above. In contrast, a strong negative correlation between these shocks is required in order to induce a negative correlation between the labor and capital steady-state gaps, and hence generate fast convergence to steady-state, as again discussed above.

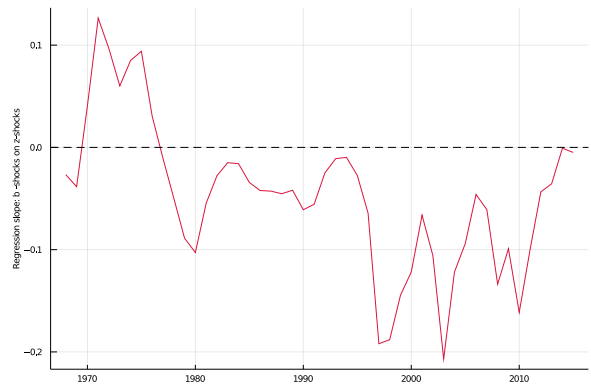
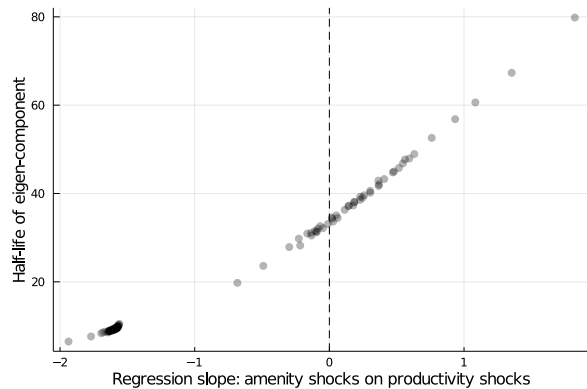
In the right panel of Figure 5, we display the slope coefficients from regressions of the empirical productivity shocks (\mathbf{z}_t) on the empirical amenity shocks (\mathbf{b}_t) for each year of our sample period, as recovered from inverting the model using our generalization of dynamic exact-hat algebra in Proposition 2 above. We find that the correlation across locations between productivity and amenity shocks is positive up to 1975, which implies slow adjustment in response to these shocks in the opening decade of our sample period, and is consistent with the initial increase in gap of the labor state variables from steady-state shown in Figure 3 above. This opening decade includes substantial structural change in manufacturing and the first oil price shock. From 1975 onwards, the correlation across locations between productivity and amenity shocks becomes negative, implying faster adjustment in response to these shocks, before diminishing to close to zero by the

end of our sample period. This pattern of results is consistent with our findings for predicted population growth above, in which we find that the explanatory power of convergence towards an initial steady-state with unchanged fundamentals is even stronger from 1975 onwards.

Figure 5: Half-lives of Convergence to Steady-State and the Regression Slope Coefficient Between the Productivity and Amenity Shocks

(a) Half-Lives and the Regression Slope Coefficient Between the Productivity and Amenity Shocks ($\tilde{\mathbf{f}}_{(h)} = \mathbf{R}^{-1}\mathbf{u}_h$) for each Eigen-shock in 2000

(b) Regression Slope Coefficient Between the Empirical Productivity and Amenity Shocks ($\mathbf{z}_t, \mathbf{b}_t$) over Time



In the left panel, half-life on the vertical axis corresponds to the time in years for the state variables to converge half of the way towards steady-state for an eigen-shock in 2000, for which the initial impact of the shock to productivity and amenities on the state variables ($\mathbf{R}\tilde{\mathbf{f}}_{(h)}$) corresponds to an eigenvector (\mathbf{u}_h) of the transition matrix (\mathbf{P}), as determined by the corresponding eigenvalue (λ_h); the horizontal axis shows slope coefficients from regressions across locations of the productivity shocks ($\tilde{\mathbf{z}}_{(h)}$) on the amenity shocks ($\tilde{\mathbf{b}}_{(h)}$) for each eigen-shock in 2000 ($\tilde{\mathbf{f}}_{(h)} = \mathbf{R}^{-1}\tilde{\mathbf{u}}_h$); each dot corresponds to a different eigencomponent in 2000; In the right panel, the vertical axis shows the slope coefficients from regressions across locations of the empirical productivity shocks (\mathbf{z}_t) on the empirical amenity shocks (\mathbf{b}_t) in each year.

Comparative Statics of the Speed of Convergence We now use our spectral analysis to evaluate the comparative statics of the speed of convergence to steady-state with respect to changes in the structural parameters of the model. Undertaking these comparative statics in the non-linear model is challenging, because the speed of convergence to steady-state depends on the incidence of the productivity and amenity shocks across the labor and capital state variables in each location. As a result, to fully characterize the impact of changes in model parameters on the speed of convergence in the non-linear model, one needs to undertake counterfactuals for the economy's transition path in response to the set of all possible productivity and amenity shocks, which is not well defined.

In contrast, our spectral analysis has two key properties. First, the set of all possible productivity and amenity shocks is spanned by the set of eigen-shocks, which is well defined. Second, we have a closed-form solution for the impact matrix (\mathbf{R}) and transition matrix (\mathbf{P}) in terms of

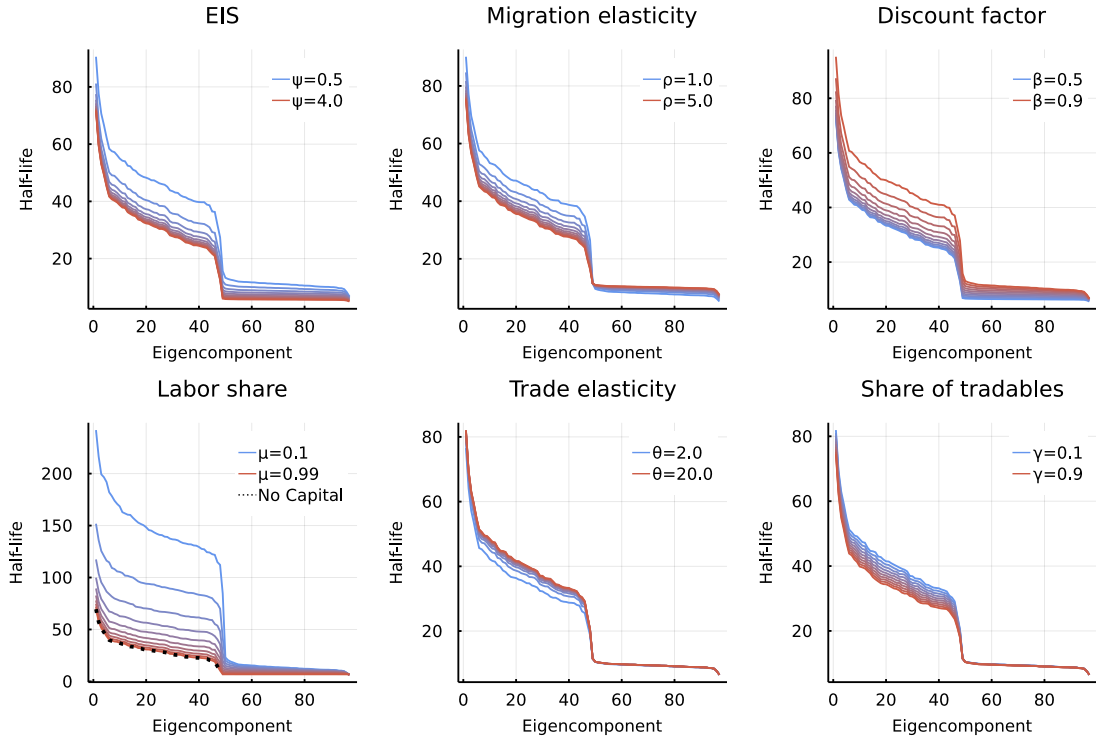
the observed data (S , T , D , E) and structural parameters $\{\psi, \theta, \beta, \rho, \mu, \delta\}$. Therefore, for any alternative model parameters, we can immediately solve for the entire spectrum of eigenvalues—and, correspondingly, the half-lives—associated with the eigen-shocks using the observed data. Because the eigen-shocks span all possible empirical productivity and amenity shocks, understanding how parameters affect the entire spectrum of half-lives translates into an analytically sharp understanding of how convergence rates are affected by model parameters.

In Figure 6, we display the half-lives of convergence to steady-state across the entire spectrum of eigen-shocks for different values of model parameters. Each panel varies the noted parameter, holding constant the other parameters at their baseline values. On the vertical axis, we display the half-life of convergence to steady-state. On the horizontal axis, we rank the eigen-shocks in terms of decreasing half-lives of convergence to steady-state for our baseline parameter values.

In the top-left panel, a lower intertemporal elasticity of substitution (ψ) implies a longer half-life (slower convergence), because consumption becomes less substitutable across time for landlords, which reduces their willingness to respond to investment opportunities. In the top-middle panel, a lower migration elasticity (ρ) also implies a longer half-life (slower convergence), because it increases the dispersion of idiosyncratic tastes, which leads workers to become less responsive to differences in the expected value of living in each location. In the top-right panel, a higher discount factor (β) also implies a longer half-life (slower convergence), because in making dynamic migration and investment decisions, both workers and landlords are less willing to sacrifice current for future utility.

In the bottom-right panel, we vary the share of expenditure on the single tradeable sector relative to the single non-tradeable sector. A lower share of tradeables (γ) implies a longer half-life (slower convergence), because it makes the impact of shocks more concentrated locally, which requires greater labor and capital reallocation between locations. In the bottom-middle panel, a higher trade elasticity (θ) implies a longer half-life (slower convergence), because it increases the responsiveness of production and consumption in the static trade model, and hence requires greater reallocation of capital and labor across locations. In the bottom-left panel, we find that a lower labor share (μ) implies a longer half-life (slower convergence), because it implies a greater role for endogenous capital accumulation, which magnifies the impact of productivity and amenity shocks, and hence requires a greater length of time for adjustment to occur.

Figure 6: Half-lives of Convergence to Steady-State for Alternative Parameter Values



Note: Half-lives of convergence to steady-state for each eigen-shock for alternative parameter values in 2000; vertical axis shows half-life in years; horizontal axis shows the rank of the eigen-shocks in terms of their half-lives for our baseline parameter values (with one the highest half life); each panel varies the noted parameter, holding constant the other parameters at their baseline values; the blue and red solid lines denote the lower and upper range of the parameter values considered, respectively; each of the other eight lines in between varies the parameters uniformly within the stated range; the thick black dotted line in the bottom-left panel displays half-lives for the special case of our model without capital, which corresponds to the limiting case in which μ converges to one.

In this bottom-left panel, we also show the half-lives of convergence to steady-state for the special case of our model with no capital using the black dotted line, which corresponds to the limiting case in which the labor share converges to one. In this special case, we only have N state variables and eigen-shocks, compared to $2N$ state variables and eigen-shocks in the general model. For these N eigen-shocks in common to both models, we find that the model with both labor and capital dynamics has slower convergence to steady-state (longer half lives) than the model with no capital, because of the role of endogenous capital accumulation in magnifying the impact of productivity and amenity shocks.

However, the model with both labor and capital dynamics has an additional N state variables and eigen-shocks, which provide an additional margin of adjustment relative to the model with no capital. As shown in the bottom-left panel, these additional N eigen-shocks are characterized by more rapid convergence to steady-state than those in common to both models. Therefore, any given empirical productivity and amenity shocks can have generate either more or less rapid adjustment in the model with both capital and labor than in the model with no capital, depending

on the loadings of these empirical shocks across the entire spectrum of eigen-shocks.

Again these findings highlight the role of the interaction between the capital and labor adjustment margins in shaping the persistent and heterogeneous effects of local labor market shocks.

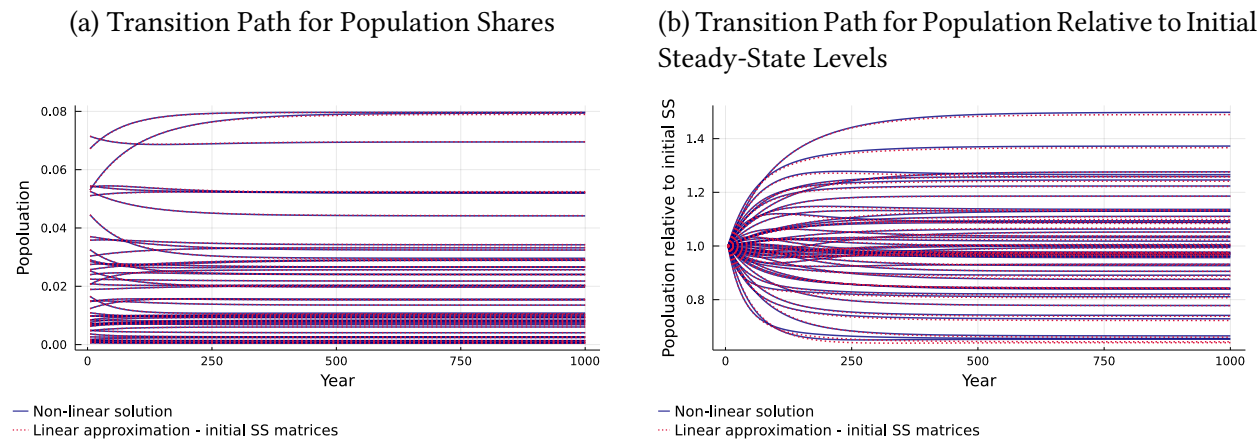
Comparing the Linearized and Non-linear Models We have so far used our spectral analysis to provide an analytical characterization of the speed of convergence to steady-state and the interaction between the capital and labor adjustment margins. While a caveat is that these analytical results are based on a linearization that is only exact for small shocks, we now show that this linearization provides a good approximation to the transition path of the non-linear model for empirically-reasonable shocks, such as decadal changes in relative productivity.

We first use our inversion of the non-linear model from our generalization of dynamic exact-hat algebra in Proposition 2 to recover the implied empirical distribution of productivities, amenities, trade costs and migration costs, as discussed in Section B.9 of the online appendix. We next solve for the steady-state of the non-linear model implied by the 1990 values of these fundamentals $\{z_i, b_i, \tau_{ni}, \kappa_{ni}\}$. Starting from this steady-state, we then undertake counterfactuals for the economy's transition path in response to the empirical distribution of productivity shocks from 1990-2000, which includes substantial changes in relative productivity ranging from around -30 to 30 percent. We solve for this transition path in both the non-linear model using Proposition 2 and the linearized model using Proposition 3.

In Figure 7, we show the economy's transition path for population shares (left panel) and population relative to the initial steady-state (right panel) for each US state. In both panels, the solid blue line denotes the non-linear model, and the red dashed-line corresponds to the linearized model. We find that the two sets of predictions track one another relatively closely along the transition path of more than one hundred years, consistent with the linearized model providing a good approximation to the solution of the non-linear model. This approximation is somewhat better for population shares (left panel) than for population relative to the initial steady-state (right panel), but remains close in both cases. We find a similar pattern of results for the capital stock and for the response of both state variables to amenity shocks.¹⁶

¹⁶While omitted in the interests of brevity, we also find a close relationship between the predictions of our linearization and the non-linear model solution for changes in steady-states, with a regression slope coefficient of 1.003 and a coefficient of correlation of 0.999. Following the approach of Kleinman et al. (2020) for a static trade model, we can also derive analytical bounds for the quality of the approximation for changes in steady-state.

Figure 7: Transition Path Predictions of Our Linearization and the Full Non-Linear Model Solution for Counterfactual Changes in Productivity



Note: We first our generalization of dynamic exact-hat algebra in Proposition 2 to recover the empirical distribution of productivities, amenities, trade costs and migration costs. We next solve for the steady-state in the full non-linear model implied by the 1990 values of these fundamentals. Starting from this steady-state, we then undertake counterfactuals for the economy’s transition path in response to the empirical distribution of productivity shocks from 1990-2000. We solve for the economy’s transition path in both the non-linear model using Proposition 2 and the linearized model using Proposition 3.

5.5 Multi-sector Quantitative Analysis

In a final empirical exercise, we implement our multi-sector extension with region-sector specific capital from Section 4.3 above, using our region-sector data from 1999-2015. We again use our spectral analysis to provide an analytical characterization of the economy’s transition path.

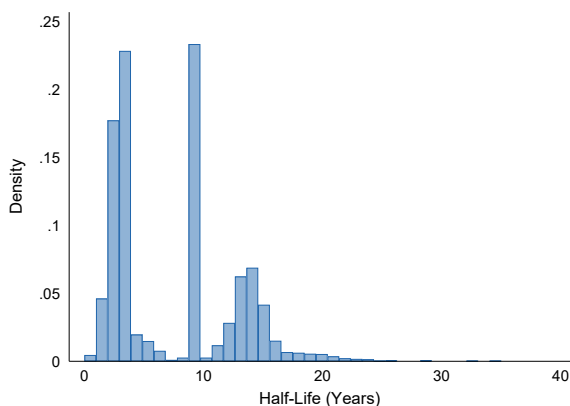
We begin by using our generalization of Proposition 5 in the multi-sector model to compute half-lives of convergence towards steady-state for shocks to productivity or amenities for which the initial impact on the state variables ($\mathbf{R}\tilde{\mathbf{f}}$) corresponds to an eigenvector (\mathbf{u}_k) of the transition matrix (\mathbf{P}). In Figure 8, we display the distribution of these half-lives across eigenvectors of the transition matrix in the year 2000. We find more rapid convergence to steady-state in our multi-sector extension, with an average half-life of 7 years and a maximum half-life of 35 years (compared to around 20 and 85 years in our baseline single-sector model). This finding is driven by the property of the region-sector migration matrices that flows of people between sectors within states are larger than those between states. A key implication is that in the persistence of local labor market shocks depends on whether they induce reallocation across industries within the same location or reallocation across different locations.

In the right panel of Figure 8, we show that capital and labor dynamics again interact with one another to shape the speed of convergence towards steady-state in the multi-sector model. On the vertical axis, we display the half-life of convergence to steady-state (in years) for each eigen-

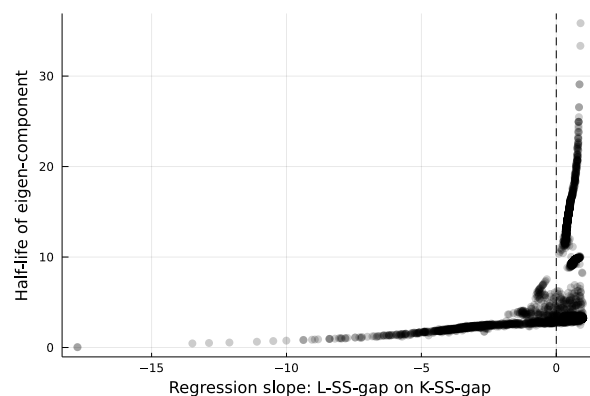
shock. On the horizontal axis, we display the regression slope coefficient between the gaps from steady-state for labor and capital for each eigen-shock. As for the single-sector model above, we find a strong, positive and non-linear relationship between the half-life of convergence to steady-state and the correlation between the gaps from steady-state for the two state variables. Although for brevity we display results for the year 2000, we again find the same pattern of results for each year of our sample period.¹⁷

Figure 8: Half-lives for Convergence Towards Steady-State in the Multi-Sector Model

(a) Histogram of Half-lives for Shocks to Productivity and Amenities that Correspond to Eigenvectors of the Transition Matrix (P) in 2000



(b) Relationship Between Half-Lives of Convergence Towards Steady-State and the Correlation between the Initial Gaps of the Labor and Capital State Variables from Steady-State in 2000



Note: Half-life corresponds to the time in years for the state variables to converge half of the way towards steady-state for an eigen-shock, for which the initial impact of the shock to productivity and amenities on the state variables ($R\tilde{f}$) corresponds to an eigenvector (u_h) of the transition matrix (P) in the multi-sector model; left panel shows the distribution of half-lives across eigencomponents of the transition matrix in 2000 in the multi-sector model; right panel plots these half-lives of convergence to steady-state for the year 2000 in the multi-sector model against the slope coefficients from regressions across location-sectors of the labor gap ($\tilde{\ell}_h$) from steady-state on the capital gap (\tilde{k}_h) from steady-state for each eigen-shock in 2000 ($u_h = R\tilde{f}_{(h)}$).

Therefore, we find a similar pattern of results for the multi-sector model as for the single-sector model above, with slow convergence to steady-state, and an important role for capital accumulation in shaping the persistent and heterogeneous impact of local labor market shocks.

6 Conclusions

A classic question in economics is the response of economic activity to local shocks. In general, this response can be gradual, because of investments in capital structures and migration frictions.

¹⁷Under our assumption of no international migration, the deviation of labor from steady-state is zero for foreign countries in our multi-sector model. Therefore, they adjust to fundamental shocks through capital accumulation alone, which is responsible for the mass of eigen-shocks with intermediate half-lives in both panels of Figure 8.

However, a key challenge in modeling these dynamics is that agents' forward-looking investment and migration decisions depend on one another in *all locations in all future time periods*.

We make four main contributions for this research question. First, we develop a tractable dynamic spatial model with forward-looking investment and migration, and provide an analytical characterization of the existence and uniqueness of the steady-state equilibrium. Second, we generalize existing dynamic exact-hat algebra results for dynamic discrete choice migration models to incorporate forward-looking investment, which allows us to undertake counterfactuals using the observed endogenous variables to capture unobserved fundamentals. Third, we linearize the model to provide an analytical characterization of the economy's transition path. Fourth, we use our framework to understand the observed decline in rates of income convergence across U.S. states since the early 1960s. Our framework is particularly well suited for this empirical application, because it allows for roles of initial conditions and shocks to fundamentals, and both capital and labor dynamics shape the speed of convergence to steady-state.

We provide evidence of slow rates of convergence to steady-state, such that individual U.S. states can remain persistently above or below steady-state for decades. As early as the 1960s, the Rust Belt states of the North-East and Mid-West were substantially above their steady-state, whereas the Sun Belt states of the South and West were markedly below their steady-state. By the end of our sample period in 2015, we find that U.S. states are closer to their steady-states than they were during the prior five decades. As a result, much of the observed decline in the rate of regional income convergence since the early 1960s is explained by initial conditions, rather than by shocks to fundamentals, such as productivity and amenities. We find that both capital accumulation and migration are important for this gradual adjustment. Of the two, capital accumulation is more important for the convergence in income per capita, whereas migration is required to match the observed changes in the population distribution across U.S. states.

We derive a closed-form solution for the economy's transition path in the linearized model in terms of an impact matrix, which captures the initial impact of shocks to fundamentals on the state variables, and a transition matrix, which governs the updating of the state variables from one period to the next. Both the transition and impact matrix can be recovered from the observed data on trade and migration shares and the structural parameters of the model. We show that the economy's response to shocks to fundamentals can be characterized in terms eigen-shocks, defined as shocks to productivity or amenities such that the initial impact on the capital and labor state variables corresponds to an eigenvector of the transition matrix. We demonstrate that the speed of convergence to steady-state for an eigen-shock depends solely on the associated eigenvalue of the transition matrix. We establish that any empirical productivity or amenity shock can be expressed as a linear combination of the eigen-shocks. Furthermore, the weights in these linear combinations can be recovered from linear projections (regressions) of the empirical

shocks on the eigen-shocks.

We use our spectral analysis to show that the economy’s transition dynamics are shaped by a powerful interaction between investment and migration decisions. Convergence towards steady-state is slow when the gaps of capital and labor from steady-state are positively correlated across locations, such that capital and labor in each location tend to either be both above or both below steady-state. In contrast convergence towards steady-state is fast when these gaps are negatively correlated across locations, such that capital tends to be above steady-state when labor is below steady-state, or vice versa. This interaction arises through the marginal products of capital and labor in the production technology. When capital is above steady-state, this raises the marginal productivity of labor, which dampens the downward adjustment of labor. Similarly, when labor is above steady-state, this raises the marginal productivity of capital, which retards the downwards adjustment of capital. These findings emphasize the role played by endogenous capital accumulation in the persistent and heterogeneous impact of local labor market shocks.

More broadly, in many applications in economics, the convergence properties of the model typically depend only on the eigenvalue with the largest absolute value, because this determines the dynamic properties of the system over long time horizons. In contrast, in our dynamic spatial model, the dynamics of the spatial distribution of economic activity depends on the entire spectrum of eigenvalues and eigenvectors, because the eigen-shocks provide a basis for the space of possible productivity and amenity shocks. Although we use our spectral analysis to characterize the dynamics of the spatial distribution of economic activity, our approach is applicable to other dynamic settings with high-dimensional state spaces.

References

- AHLFELDT, G., S. REDDING, D. STURM, AND N. WOLF (2015): “The Economics of Density: Evidence from the Berlin Wall,” *Econometrica*, 83, 2127–2189.
- ALDER, S., D. LEGAKOS, AND L. OHANIAN (2019): “Labor Market Conflict and the Decline of the Rust Belt,” Boston University, mimeograph.
- ALLEN, T. AND C. ARKOLAKIS (2014): “Trade and the Topography of the Spatial Economy,” *Quarterly Journal of Economics*, 129, 1085–1140.
- ALLEN, T., C. ARKOLAKIS, AND X. LI (2017): “Optimal City Structure,” Yale University, mimeograph.
- (2020): “On the Equilibrium Properties of Network Models with Heterogeneous Agents,” *NBER Working Paper*, 27837.
- ALLEN, T. AND D. DONALDSON (2020): “The Geography of Path Dependence,” Dartmouth College, mimeograph.
- ALVAREZ, F. (2017): “Capital Accumulation and International Trade,” *Journal of Monetary Economics*, 91, 1–18.
- ANDERSON, J. E., M. LARCH, AND Y. V. YOTOV (2015): “Growth and Trade with Frictions: A Structural Estimation Framework,” *NBER Working Paper*, 21377.

- ANGELETOS, G.-M. (2007): “Uninsured Idiosyncratic Investment Risk and Aggregate Saving,” *Review of Economic Dynamics*, 10, 1–30.
- ARMINGTON, P. S. (1969): “A Theory of Demand for Products Distinguished by Place of Production,” *IMF Staff Papers*, 16, 159–178.
- ARTUÇ, E., S. CHAUDHURI, AND J. McLAREN (2010): “Trade Shocks and Labor Adjustment: A Structural Empirical Approach,” *American Economic Review*, 100, 1008–45.
- AUTOR, D., D. DORN, AND G. H. HANSON (2013): “The China Syndrome: Local Labor Market Effects of Import Competition in the United States,” *American Economic Review*, 103, 2121–68.
- BARRO, R. J. AND X. SALA-I-MARTIN (1992): “Convergence,” *Journal of Political Economy*, 100, 223–251.
- BLANCHARD, O. J. AND C. M. KAHN (1980): “The Solution of Linear Difference Models Under Rational Expectations,” *Econometrica*, 48, 1305–11.
- BLANCHARD, O. J. AND L. F. KATZ (1992): “Regional Evolutions,” *Brookings Papers on Economic Activity*, 1, 1–61.
- BRYAN, G. AND M. MORTEN (2019): “The Aggregate Productivity Effects of Internal Migration: Evidence from Indonesia,” *Journal of Political Economy*, 127, 2229–2268.
- CALIENDO, L., M. DVORKIN, AND F. PARRO (2019): “Trade and Labor Market Dynamics: General Equilibrium Analysis of the China Trade Shock,” *Econometrica*, 87, 741–835.
- CALIENDO, L. AND F. PARRO (2015): “Estimates of the Trade and Welfare Effects of NAFTA,” *Review of Economic Studies*, 82, 1–44.
- (2020): “The Quantitative Effects of Trade Policy on Industrial and Labor Location,” Yale University, mimeograph.
- CALIENDO, L., F. PARRO, E. ROSSI-HANSBERG, AND P.-D. SARTE (2018): “The Impact of Regional and Sectoral Productivity Changes on the U.S. Economy,” *Review of Economic Studies*, 85, 2042–2096.
- CASSEY, A. J. (2009): “State Export Data: Origin of Movement Versus Origin of Production,” *Journal of Economic and Social Measurement*, 34, 241–268.
- COSTINOT, A., D. DONALDSON, AND I. KOMUNJER (2012): “What Goods Do Countries Trade? A Quantitative Exploration of Ricardo’s Ideas,” *Review of Economic Studies*, 79, 581–608.
- COSTINOT, A. AND A. RODRÍGUEZ-CLARE (2014): “Trade Theory with Numbers: Quantifying the Consequences of Globalization,” in *Handbook of International Economics*, ed. by E. Helpman, K. Rogoff, and G. Gopinath, Amsterdam: Elsevier North Holland, vol. 4, 197–262.
- DESMET, K., D. K. NAGY, AND E. ROSSI-HANSBERG (2018): “The Geography of Development,” *Journal of Political Economy*, 126, 903–983.
- DESMET, K. AND E. ROSSI-HANSBERG (2014): “Spatial Development,” *American Economic Review*, 104, 1211–1243.
- DIX-CARNEIRO, R. AND B. K. KOVAK (2017): “Trade Liberalization and Regional Dynamics,” *American Economic Review*, 107, 1908–2946.
- DURANTON, G. AND D. PUGA (2004): “Micro-Foundations of Urban Agglomeration Economies,” in *Handbook of Regional and Urban Economics*, ed. by J. V. Henderson and J.-F. Thisse, Amsterdam: Elsevier, vol. 4, 2063–2117.
- EATON, J., S. KORTUM, B. NEIMAN, AND J. ROMALIS (2016): “Trade and the Global Recession,” *American Economic Review*, 106, 3401–38.
- ERIKSSON, K., K. N. RUSS, J. C. SHAMBAUGH, AND M. XU (2019): “Trade Shocks and the Shifting Landscape of U.S. Manufacturing,” *NBER Working Paper*, 25646.
- FAJGELBAUM, P. D., E. MORALES, J. C. SUÁREZ-SERRATO, AND O. ZIDAR (2019): “State Taxes and Spatial Misallocation,” *Review of Economic Studies*, 86, 333–376.
- FUJITA, M., P. KRUGMAN, AND A. J. VENABLES (1999): *The Spatial Economy: Cities, Regions, and International Trade*, Cambridge MA: MIT Press.
- GANONG, P. AND D. SHOAG (2017): “Why has Regional Income Convergence in the US Declined,” *Journal of Urban Economics*, 102, 76–90.

- GLAESER, E. L. AND J. GYOURKO (2005): “Urban Decline and Durable Housing,” *Journal of Political Economy*, 113, 345–375.
- GREANEY, B. (2020): “The Distributional Effects of Uneven Regional Growth,” University of Washington, mimeograph.
- HELPMAN, E. (1998): “The Size of Regions,” in *Topics in Public Economics: Theoretical and Applied Analysis*, ed. by D. Pines, E. Sadka, and I. Zilcha, Cambridge: Cambridge University Press, 33–54.
- HYATT, H., E. MCENTARFER, K. UEDA, AND A. ZHANG (2018): “Interstate Migration and Employer-to-Employer Transitions in the United States: New Evidence from Administrative Records Data,” *Demography*, 55, 2161–2180.
- KAPLAN, G. AND S. SCHULHOFER-WOHL (2017): “Understanding the Long-run Decline in Interstate Migration,” *International Economic Review*, 58, 57–94.
- KAPLAN, G. AND G. L. VIOLANTE (2014): “A Model of the Consumption Response to Fiscal Stimulus Payments,” *Econometrica*, 82, 1199–1239.
- KIM, S. (1995): “Expansion of Markets and the Geographic Distribution of Economic Activities: The Trends in U.S. Regional Manufacturing Structure 1860-1987,” *Quarterly Journal of Economics*, 881–908.
- KLEINMAN, B., E. LIU, AND S. J. REDDING (2020): “International Friends and Enemies,” *NBER Working Paper*, 27587.
- KRUGMAN, P. (1991): “Increasing Returns and Economic Geography,” *Journal of Political Economy*, 99, 483–499.
- KRUGMAN, P. AND A. VENABLES (1995): “Globalization and the Inequality of Nations,” *Quarterly Journal of Economics*, 110, 857–880.
- MITCHENER, K. J. AND I. W. MCLEAN (1999): “U.S. Regional Growth and Convergence 1880-1980,” *Journal of Economic History*, 59, 1016–1042.
- MOLL, B. (2014): “Productivity Losses from Financial Frictions: Can Self-Financing Undo Capital Misallocation,” *American Economic Review*, 104, 3186–3221.
- MOLLOY, R., C. L. SMITH, AND A. WOZNIAK (2011): “Internal Migration in the United States,” *Journal of Economic Perspectives*, 25, 173–196.
- MONTE, F., S. REDDING, AND E. ROSSI-HANSBERG (2018): “Commuting, Migration and Local Employment Elasticities,” *American Economic Review*, 108, 3855–3890.
- PETERS, M. (2019): “Market Size and Spatial Growth: Evidence from Germany’s Post-War Population Expulsions,” Yale University, mimeograph.
- RAMONDO, N., A. RODRÍGUEZ-CLARE, AND M. SABORÍP-RODRÍGUEZ (2016): “Trade, Domestic Frictions and Scale Effects,” *American Economic Review*, 106, 3159–3184.
- RAVIKUMAR, B., A. M. SANTACREU, AND M. SPOSI (2019): “Capital Accumulation and Dynamic Gains from Trade,” *Journal of International Economics*, C, 93–110.
- REDDING, S. J. (2016): “Goods Trade, Factor Mobility and Welfare,” *Journal of International Economics*, 101, 148–167.
- (2020): “Trade and Geography,” in *Handbook of Regional and Urban Economics*, ed. by G. Gopinath, E. Helpman, and K. Rogoff, Amsterdam: Elsevier, forthcoming.
- REDDING, S. J. AND E. ROSSI-HANSBERG (2017): “Quantitative Spatial Models,” *Annual Review of Economics*, 9, 21–58.
- REDDING, S. J. AND D. M. STURM (2008): “The Costs of Remoteness: Evidence from German Division and Reunification,” *American Economic Review*, 98, 1766–1797.
- UHLIG, H. (1999): “A Toolkit for Analyzing Non-linear Dynamic Stochastic Models Easily,” in *Computational Methods for the Study of Dynamic Economies*, ed. by R. Marimon and A. Scott, New York: Oxford University Press, 30–61.
- WALSH, C. (2019): “Firm Creation and Local Growth,” Yale University, mimeograph.